Phys 102 - MT1 - F19

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1- (25 pts) As shown in the figure a positive charge Q is uniformly distributed along a thin rod of length L.

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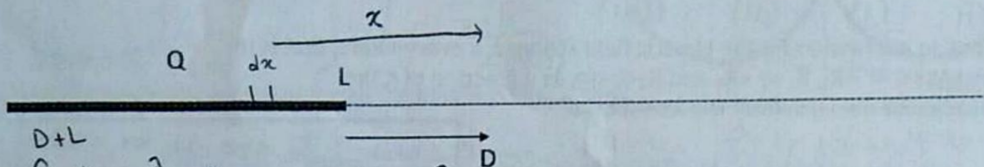
a) Calculate the electric field vector produced by a charged rod at a distance D.

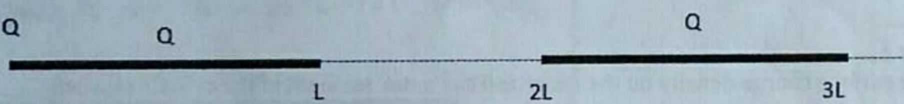


$$|\vec{E}| = \int_D^{D+L} \frac{1}{4\pi\varepsilon_0} \frac{\lambda}{x^2}\, dx \quad \text{with } \lambda := \frac{Q}{L}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left(-\frac{1}{x}\right)\Bigg|_{x=D}^{D+L} = \frac{\lambda}{4\pi\varepsilon_0}\left(\frac{1}{D} - \frac{1}{D+L}\right)$$

$$\vec{E} = |\vec{E}|\,\hat{x}$$

b) The second uniformly charged rod is placed on the x axis (see the figure). The second rod also has positive charge Q distributed uniformly. Calculate the magnitude of the electrostatic force applied on the second rod.



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$$|\vec{F}| = \int |\vec{E}|\lambda\, dx$$

$$= \frac{\lambda^2}{4\pi\varepsilon_0} \int_L^{2L} \left(\frac{1}{x} - \frac{1}{x+L}\right) dx$$

$$= \frac{\lambda^2}{4\pi\varepsilon_0}\left(\log|x|\Big|_{x=L}^{2L} - \log|x+L|\Big|_{x=L}^{2L}\right)$$

$$= \frac{\lambda^2}{4\pi\varepsilon_0}\Big(\underbrace{\log|2L| - \log|L|}_{=\log\frac{|2L|}{|L|}} \underbrace{- \log|3L| + \log|2L|}_{-\log\frac{|3L|}{|2L|}}\Big) = \frac{\lambda^2}{4\pi\varepsilon_0}\left(\log 2 - \log\frac{3}{2}\right)$$

$$= \frac{\lambda^2}{4\pi\varepsilon_0} \log\frac{4}{3}$$

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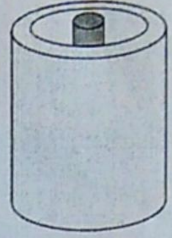
2- (25 pts) Consider a coaxial cable with an inner conductor shielded by a conductor layer. Assume the inner conductor is an infinite line charge of uniform density λ, while the surrounding conductor is a neutral hollow cylindrical shell with inner Radius R_1 and outer radius R_2 (see the figure).



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(a) Find an expression for the electric field strength E everywhere, that is in the regions (I) $0 < r < R_1$, (II) $R_1 < r < R_2$ and (III) $R_2 < r < \infty$, as a function of r, the perpendicular distance from the line charge.

By sym., on a coaxial cylindrical surface S $|\vec{E}|$ is constant and the direction is parallel to the surface element.

Hence Gauss law becomes

Note: S should be closed, but the upper and lower parts do not contribute because they are orthogonal to $\vec{E}$.

$$\oint_S \vec{E}\cdot d\vec{A} = \oint_S |\vec{E}|\,dA = |\vec{E}| \oint_S dA = |\vec{E}|\,\text{Area}(S) = \frac{Q_{encl}}{\varepsilon_0}$$

1) $|\vec{E}|\,2\pi r l = \dfrac{\lambda l}{\varepsilon_0}$ where r is the radius, l is the length of the cylinder

$\Rightarrow |\vec{E}| = \dfrac{\lambda}{2\pi r \varepsilon_0}$

(b) Find the surface charge density on the inner and the outer surfaces of the cylindrical shell.

2) $|\vec{E}| = 0$ inside the conductor

3) Same procedure of (1) applies here because conductor cylinder is neutral.

$|\vec{E}| = \dfrac{\lambda}{2\pi r \varepsilon_0}$

) In order to have $|\vec{E}| = 0$ inside the conductor, electrons move to the inner or outer surface creating an electric field which kills the electric field that is created by the charged cable.

By Gauss law in (2) on the inner surface we should have the same amount of opposite charges per unit length. But the cylinder is a 2-dim surface, hence it has a surface charge density $\sigma_{in} = \dfrac{-\lambda}{2\pi R_1}$ | For neutrality, $\sigma_{out} = \dfrac{\lambda}{2\pi R_2}$

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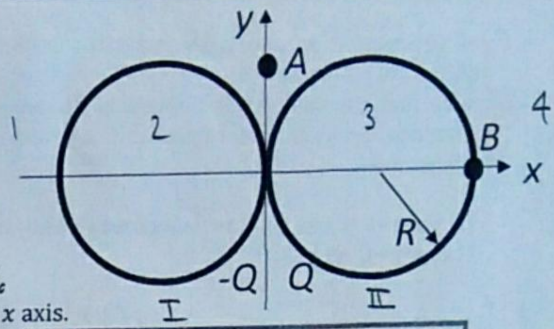
Gauss law)

3- (25 pts) Two empty spherical shells of radius R are centered symmetrically around the origin, and have uniform charges $\pm Q$ as in the figure.



For a sphere of radius $r>R$: $|\vec{E}| = \dfrac{\pm Q}{4\pi r^2 \varepsilon_0}$ $\Rightarrow$ $V = \pm \dfrac{Q}{4\pi r \varepsilon_0}$

$r<R$: $|\vec{E}| = 0$ $\Rightarrow$ $V = \pm \dfrac{Q}{4\pi R \varepsilon_0}$

a) Find the electrical potential for all the points on the x axis.

Note: $\hat{x}$ is parallel to $\hat{r}$

1) $x \leq -2R$, $V_I + V_{II} = \dfrac{-Q}{4\pi(x-R)\varepsilon_0} + \dfrac{Q}{4\pi(x+R)\varepsilon_0}$

2) $-2R < x \leq 0$, $V_I + V_{II} = \dfrac{-Q}{4\pi R \varepsilon_0} + \dfrac{Q}{4\pi(x+R)\varepsilon_0}$

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3) $0 < x < 2R$, $V_I + V_{II} = \dfrac{-Q}{4\pi(x+R)\varepsilon_0} + \dfrac{Q}{4\pi R \varepsilon_0}$

4) $2R \leq x$, $V_I + V_{II} = \dfrac{-Q}{4\pi(x+R)} + \dfrac{Q}{4\pi(x-R)}$

b) Draw one equipotential surface for this charge configuration, and explain your reasoning.

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yz-plane is an equipotential surface. Any point on this plane will be at the same distance from the spheres I and II. Hence, the contrubutions from $-Q$ and Q cancel each other and we have an equipotential surface with $V=0$.

c) How much work do we have to do to move a point charge q from point A at $(0, R)$ to point B at $(2R, 0)$?

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Work energy theorem:

$W = \Delta U$ where U is the potential energy. For electric force $U = qV$. Hence we have

$$W = q\Delta V = q(V_B - V_A) = q\left[\left(-\frac{Q}{4\pi(2R+R)\varepsilon_0} + \frac{Q}{4\pi(2R-R)\varepsilon_0}\right) - 0\right]$$

$$= \frac{qQ}{4\pi\varepsilon_0 R}\left(-\frac{1}{3} + 1\right) = \frac{qQ}{6\pi\varepsilon_0 R}$$

part a (4) part b

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4- (25 pts) A parallel plate capacitor is filled with a nonuniform dielectric characterized by a dielectric constant $\kappa(x) = k/x^n$, where k and n are positive constants (k has dimensions to make sure that the dielectric constant is dimensionless) and x is the distance from one plate. The distance between the capacitor plates and the area of each plate are denoted by d and A, respectively.

(i) Use the Gauss law for dielectrics to find the electric field between the plates if the plates have charges +Q and -Q. 7

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{Q_{encl}}{\varepsilon} \quad \text{where } \varepsilon := \varepsilon_0 k(x)$$

Choose S as a cylinder with upper and lower parts attached. Note that the sides do not contribute because $\vec{E} \perp d\vec{A}$. Moreover on the upper and lower parts $\vec{E} \parallel d\vec{A}$ and $|\vec{E}|$ is constant. Moreover $Q_{encl} = \sigma\tilde{A}$ where $\tilde{A}$ is the area of upper and lower parts, $\sigma := \frac{Q}{A}$. Hence $\oint_S \vec{E}\cdot d\vec{A} = |\vec{E}|\tilde{A} = \frac{\sigma\tilde{A}}{\varepsilon}$

$$\Longrightarrow |\vec{E}| = \frac{Q x^n}{A\varepsilon_0 k}$$

(ii) Find the voltage difference between the plates. 8

$$\Delta V = \int_{①}^{②} \vec{E}\cdot d\vec{l} = \int_0^d \frac{Q x^n}{A\varepsilon_0 k}\,dx$$

$$= \frac{Q}{A\varepsilon_0 k}\,\frac{x^{n+1}}{n+1}\Big|_{x=0}^{d} = \frac{Q d^{n+1}}{A\varepsilon_0 k(n+1)}$$

(iii) Calculate the capacitance of the system by dividing the dielectric to many thin layers and using capacitor connection rules for the equivalent capacitance. Compare your result with the capacitance calculation based upon the results in (i) and (ii). 10

$$\frac{1}{C_{total}} = \int \frac{1}{dC} = \int_0^d \frac{x^n dx}{Ak\varepsilon_0}$$

$$= \frac{1}{Ak\varepsilon_0}\,\frac{x^{n+1}}{n+1}\Big|_{x=0}^{d} = \frac{d^{n+1}}{Ak\varepsilon_0(n+1)}$$

$$\Longrightarrow C_{total} = \frac{A\varepsilon_0 k(n+1)}{d^{n+1}} = \frac{\Delta V}{Q}$$

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