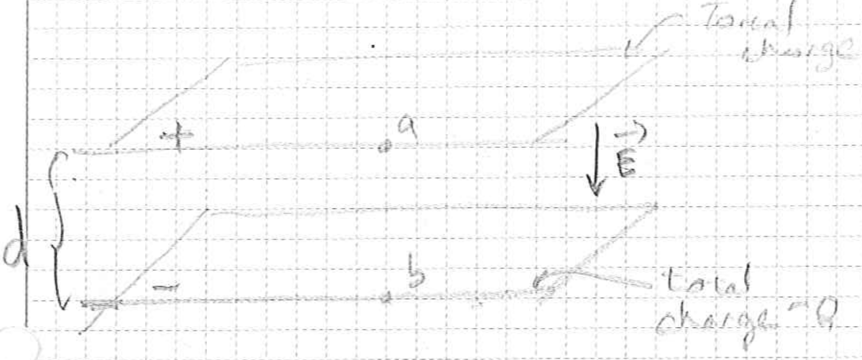
Chapter 24: Capacitance and Dielectrics

Consider parallel conducting plates each having charges $+Q$ and $-Q$.



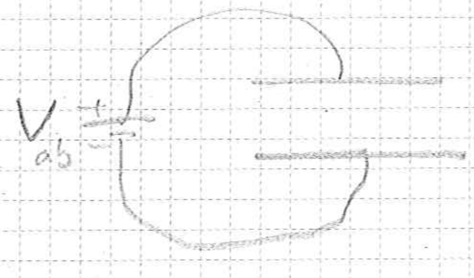
$$\vec{E} = -\frac{\sigma}{\epsilon_0}\hat{j}, \quad \sigma = \frac{Q}{A}$$

⇒ V_{ab} : potential difference between the positively charged plate and negatively charged plate is:

$$V_{ab} = \frac{\int_{y_a}^{y_b} -q_0 E\,dy}{q_0} = \frac{1}{q_0} q_0 \int_{y_a}^{y_b} -E\,dy = -E(y_b - y_a) = \boxed{Ed}$$

$$V_{ab} = \frac{\sigma}{\epsilon_0} d = \frac{Q}{A\epsilon_0} d = \frac{Q}{\left(\frac{A\epsilon_0}{d}\right)}$$ ← The total charge in the plates.

Now consider charging of the capacitor:



If A voltage V_{ab} is applied total charge $+Q$ and $-Q$ will be generated in plates.

for $A \uparrow$ larger Q
$d \downarrow$ " Q
⇒ $\frac{\epsilon_0 A}{d}$

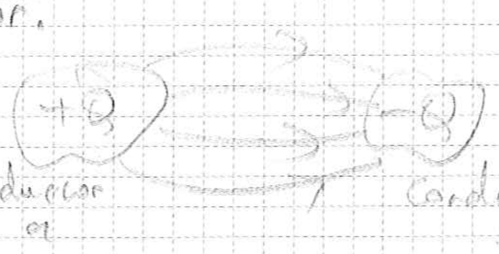
⇒ $\frac{\epsilon_0 A}{d}$ describes the ability of the parallel plates to store charge.

This is called the capacitance $\boxed{C = \frac{Q}{V_{ab}}}$

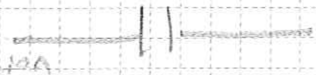
$C = \dfrac{Q}{V_{ab}}$: V_{ab}: potential difference between the positively and negatively charged plates

$+Q$ is the charge stored in $+$'ly charged plate.

In general, any two conductors separated by an insulator forms a capacitor.



$C = \dfrac{Q}{V_{ab}}$: capacitance

Circuit representation


$$C = \frac{Q}{V_{ab}}$$

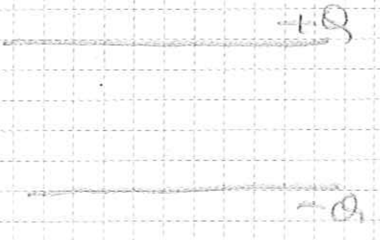
$$(\text{Farad}) = \frac{(\text{Coulomb})}{(\text{Volt})}$$

<u>Calculating Capacitance in Vacuum:</u>

Approach: Given two conductors assume a charge $+Q$ on one and $-Q$ on the other conductor. And calculate V_{ab}, potential difference b/w the $+$ and $-$ charged conductor.

$$\Rightarrow C = \frac{Q}{V_{ab}}$$

<u>Parallel Plate Capacitor</u>



$$V_{ab} = Ed = \frac{\sigma}{\epsilon_0} d$$

$$\Rightarrow C = \frac{\epsilon A}{d} \Leftarrow = \frac{Q}{A \epsilon_0} d$$

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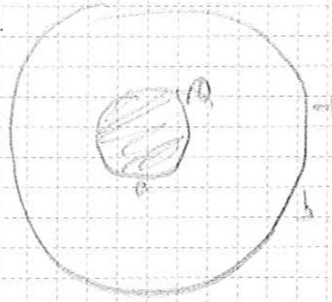
Ex 24.3: Spherical Capacitor



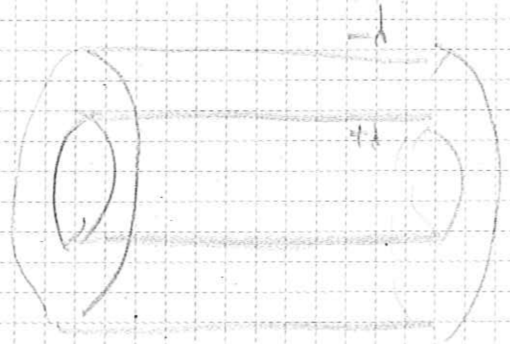
Capacitance of two concentric spherical conducting shells.

$$V_{ab} = \int_a^b \vec{E}\cdot d\vec{l} = \int_a^b \frac{1}{4\pi\epsilon_0}\cdot\frac{Q}{r^2}\,dr$$

$$= \frac{Q}{4\pi\epsilon_0}\left.-\frac{1}{r}\right|_a^b = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_a}-\frac{1}{r_b}\right)$$

$$\Rightarrow C = \frac{Q}{V_{ab}} = \frac{4\pi\epsilon_0}{\frac{1}{r_a}-\frac{1}{r_b}} = \boxed{\frac{4\pi\epsilon_0\, r_a r_b}{r_b - r_a}}$$

Ex 24.4: Cylindrical Capacitor



Long cylinder, linear charge density λ surrounded by cylindrical shell

What is the capacitance per unit length?

Consider a portion with length L:

$$2\pi r E(r) L = \frac{\lambda L}{\epsilon_0} \Rightarrow E(r) = \frac{\lambda}{2\pi\epsilon_0 r}$$

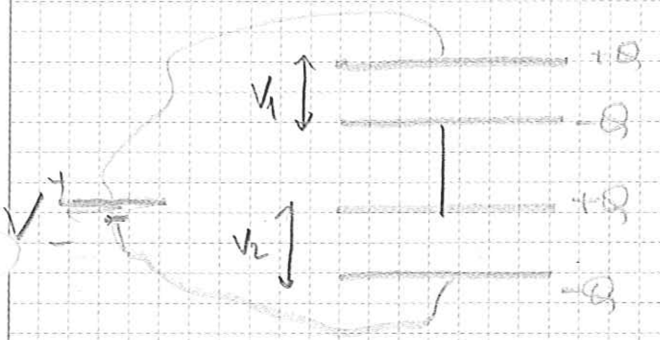
$$\Rightarrow V_{ab} = \int_a^b \frac{\lambda}{2\pi\epsilon_0 r}\,dr = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{r_b}{r_a}\right) \Rightarrow C = \frac{Q}{V_{ab}} = \frac{\lambda L}{V_{ab}}$$

$$\Rightarrow C = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{r_b}{r_a}\right)} \Rightarrow \text{Capacitance per unit length: } \frac{C}{L} = \frac{2\pi\epsilon_0}{\ln\left(\frac{r_b}{r_a}\right)}$$

Capacitors in Series and Parallel

Capacitors in Series

Suppose you have two parallel plate capacitors initially uncharged.



The capacitors are serially interconnected. When a potential difference V is applied to this system. The plates will be charged such that they have the same magnitude of charge.

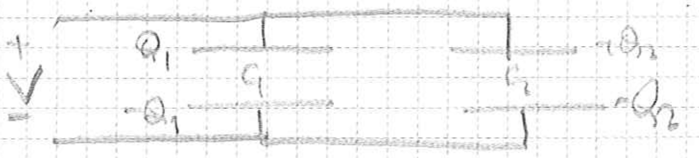
$$\Rightarrow V = V_1 + V_2 = \frac{Q}{C_1} + \frac{Q}{C_2} \Rightarrow V = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right) = \frac{Q}{C_{eq}}$$

$$\boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}}$$

For capacitors in series

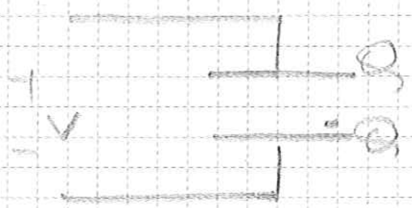
$$\boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \dots}$$

Capacitors in Parallel



In a parallel connection the potential difference of all individual capacitors is the same.

$$\Rightarrow V = \frac{Q_1}{C_1}, \quad V = \frac{Q_2}{C_2}$$



Total charge stored in the capacitors

$$Q = Q_1 + Q_2$$

$$\Rightarrow Q = V(C_1 + C_2)$$

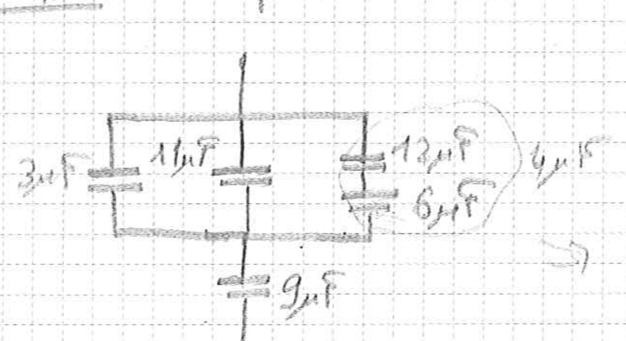
$$\Rightarrow (C_1 + C_2) = \frac{Q}{V} = C_{eq}$$

$$\Rightarrow \boxed{C_{eq} = C_1 + C_2}$$

For many capacitors in parallel:

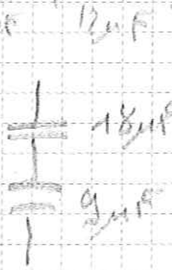
$$\boxed{C_{eq} = C_1 + C_2 + \ldots}$$

Ex 24.6: A capacitor network:

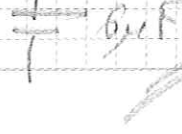


Solve the problem in pieces

$$\frac{1}{6\mu F} + \frac{1}{12\mu F} = \frac{1}{4\mu F}$$



$$\frac{1}{9\mu F} + \frac{1}{18\mu F} = \frac{1}{6\mu F}$$



$V_{ab} = V_a - V_b = \int_a^b \vec{E} \cdot d\vec{l}$, $W_{ab} = U_a - U_b = V_a - V_b$

24.3 Energy Storage in Capacitors and Electric-Field Energy

The work required to charge a capacitor gives the potential energy stored in the capacitor.

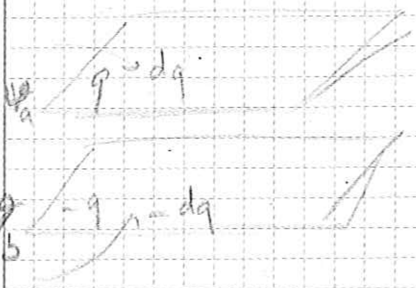
Suppose the final voltage of the capacitor is V.

$$V = \frac{Q}{C}$$

At an intermediate stage : $v = \frac{q}{C}$

Work done by the external force to add a charge dq to the capacitor:

$$dW = dq\,(v_a - 0) + (-dq)(v_b - 0) = dq\,(v_a - v_b) = dq\,v$$



$v_a - v_b = v$

$$\Rightarrow W = \int dW = \int v\,dq = \int \frac{q}{C}\,dq = \frac{1}{C}\frac{Q^2}{2} = \boxed{\frac{Q^2}{2C}}$$

work done in charging a capacitor

W will also describe the total work done by the electric field when the capacitor discharges.

If we define the potential energy of an uncharged capacitor to be equal to 0 then

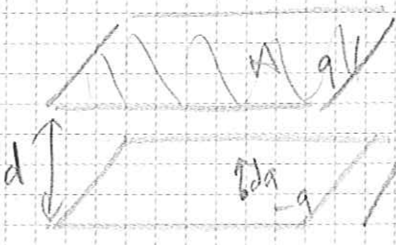
$$U = \frac{Q^2}{2C}$$, potential energy of the charged capacitor.

$$= \frac{1}{2}CV^2 = \frac{1}{2}QV$$

Electric-Field Energy

We can also imagine charging of a capacitor by moving electrons from one place to another.

In this case energy is stored in the field between the plates.



Energy density:

$$u = \frac{\frac{1}{2}CV^2}{Ad}$$

$$dW_{-\to+} = dq\,(V_- - V_+) = -dq\,V_{+-}$$

$$dW_{ext} = dq\,V_{+-}$$

$$\boxed{W_{ext} = \frac{1}{2}CV^2}$$

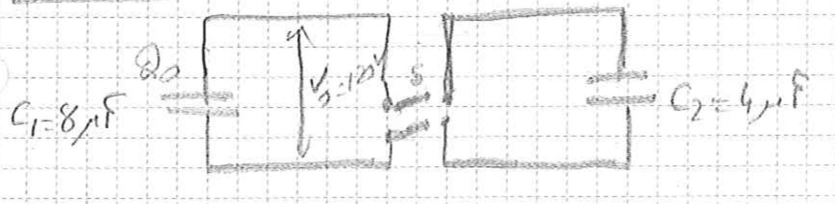
For parallel plates:

$$C = \epsilon_0 \frac{A}{d}, \quad V = Ed$$

$$\Rightarrow u = \frac{1}{2}\frac{\epsilon_0 \frac{A}{d} E^2 d^2}{Ad} = \boxed{\frac{1}{2}\epsilon_0 E^2}$$

This expression for energy density is valid for any electric field configuration in vacuum.

Ex 24.7:



C_1 is charged by connecting to a source of potential difference $V_0 = 120V$. Once C_1 is charged the source of potential difference is disconnected.

a) What is the charge Q_0 if the switch S is left open

$$Q_0 = C_1 V_0 = (8\mu F)(120V) = 960\mu C$$

b) Energy stored in C_1 if S is open

$$\frac{1}{2}C_1 V_0^2 = \frac{1}{2}(8\mu F)(120V)^2 = 0.058J$$

c) S is closed $\Rightarrow$ What is the potential difference across each capacitor, charge across each capacitor?

Total charge will be conserved

$$\Rightarrow Q_0 = Q_1 + Q_2 = C_1V + C_2V = (C_1 + C_2)V$$

$$V_1 = V_2$$

$$\Rightarrow 960\mu C = (8\mu F + 4\mu F)V \Rightarrow \boxed{V = 80V}$$

$$Q_1 = C_1V = 8\mu F \times 80V = 640\mu C$$

$$Q_2 = C_2V = 4\mu F \times 80V = 320\mu C$$

d) Total energy of the system after S is closed:

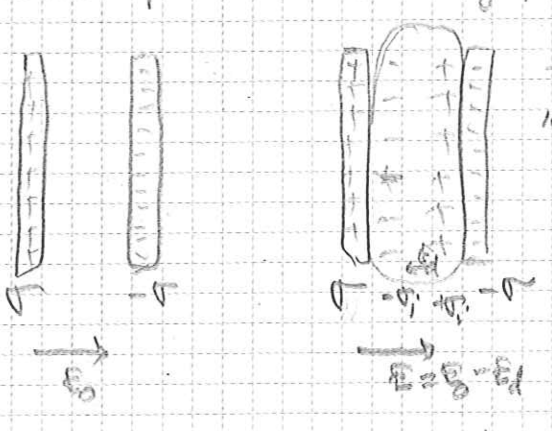
$$\left.\begin{array}{l} U_1 = \frac{1}{2}C_1V^2 \\ U_2 = \frac{1}{2}C_2V^2 \end{array}\right\} \Rightarrow U = \frac{1}{2}(8\mu F + 4\mu F)(80V)^2 = \boxed{0.038J} < \boxed{0.058J}$$

energy lost during charging of the capacitors.

24.4 Dielectrics:

Insulating (non-conducting) materials placed in capacitors.

Consider parallel conducting plates



when a dielectric material is placed between the parallel plates, surface charges are induced in the dielectric.

"Polarization"

Induced surface charges cause a decrease in the electric field intensity between the parallel plates.

To the first approximation

$$E = \frac{E_0}{K} \quad \Big\} \text{ decreased electric field intensity compared to vacuum}$$

dielectric constant.

Conductor | Dielectric
σ | $-\sigma_i$

without the dielectric:

$E_0 = \frac{\sigma}{\epsilon_0}$ with dielectric: $E = \frac{\sigma}{K\epsilon_0} = \frac{\sigma}{\epsilon} = \frac{\sigma - \sigma_i}{\epsilon_0}$

σ_i: induced charge density

$K\epsilon_0$: permittivity of the dielectric

$\frac{\sigma}{K\epsilon_0} = \frac{\sigma - \sigma_i}{\epsilon_0}$

$\Rightarrow \boxed{\sigma_i = \sigma\left(1 - \frac{1}{K}\right)}$

now $E = \frac{\sigma}{\epsilon}$ $\Rightarrow$ all the formulas for vacuum will be valid for the case of a dielectric when ϵ_0 is substituted with ϵ.

$\Rightarrow$ $C = \epsilon \frac{A}{d}$, capacitance of parallel plates

$u = \frac{1}{2}\epsilon E^2$, energy density.

Gauss's Law in Dielectrics:

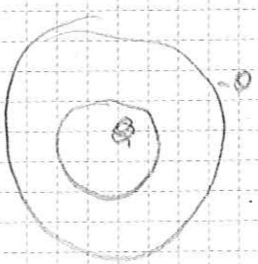
Gauss's Law is modified in dielectrics as:

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q - Q_i}{\epsilon_0} = \frac{Q_{encl,free}}{K\epsilon_0}$$

Total free charge enclosed by the Gaussian surface

Ex 24.12: Spherical capacitor with dielectric:

If the volume between spherical shells is filled with oil (dielectric constant K), what is the capacitance?



According to Gauss's Law:

$$\oint K\vec{E} \cdot d\vec{A} = K E(r) 4\pi r^2 = \frac{Q}{\epsilon_0} \Rightarrow E(r) = \frac{1}{4\pi\epsilon} \frac{Q}{r^2}$$

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$$E(r) = \frac{1}{4\pi\epsilon}\frac{Q}{r^2} \Rightarrow V_{a\to b} = \int_a^b \frac{1}{4\pi\epsilon}\frac{Q}{r^2}\,dr$$

$$= \int_{r_a}^{r_b} \frac{1}{4\pi\epsilon}\frac{Q}{r^2}\,dr = \frac{1}{4\pi\epsilon}Q\left(\frac{1}{r_a}-\frac{1}{r_b}\right)$$

$$\Rightarrow C = \frac{Q}{V} = \frac{4\pi\epsilon}{r_b - r_a} r_a r_b$$

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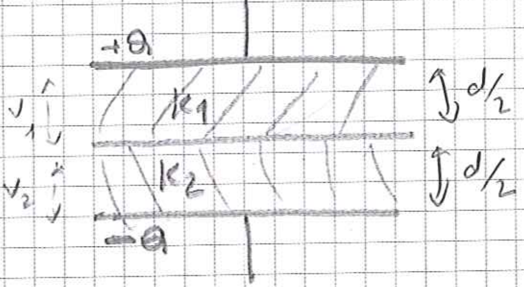
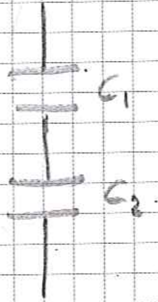
Prob 24.71:



Parallel plate capacitor, space between the plates filled with two slabs of dielectric. Each slab has thickness $d/2$.

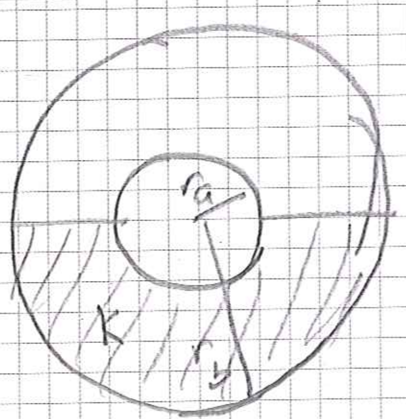
Show that $C = \frac{2\epsilon_0 A}{d}\left(\frac{K_1 K_2}{K_1+K_2}\right)$



$C_1 = \frac{2K_1 A\epsilon_0}{d}$, $C_2 = \frac{2K_2 A\epsilon_0}{d}$

$$C_{eq} = \frac{C_1 C_2}{C_1 + C_2} = \frac{2A\epsilon_0}{d}\left[\frac{K_1 K_2}{K_1+K_2}\right]$$

Prob 24.78:



Spherical capacitor.
Half of the volume between two conductors is filled with a liquid dielectric of constant K.

a) Capacitance of the half filled capacitor

b) Magnitude of $\vec{E}$ in the volume between the two conductors, for both the upper and lower halves

c) Surface density of free charges on the upper and lower halves of the inner and outer conductors.

d) Surface density of bound charge at $r=r_a$, and $r=r_b$.

e) Surface density of bound charge on the flat surface of the dielectric?

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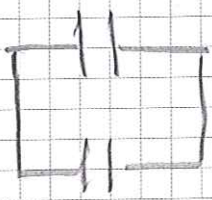
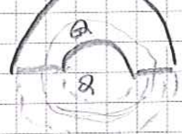
Parallel connection

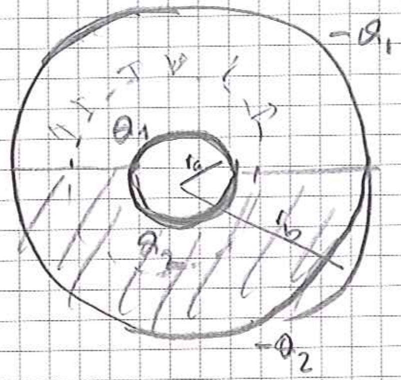
remember: $C = 4\pi\epsilon_0 \frac{r_a r_b}{(r_b - r_a)} = \frac{Q}{V}$

for a full sphere



For a half sphere: $C = \frac{Q}{2V}$





$$C_1 = \frac{4\pi\epsilon_0}{2}\frac{r_a r_b}{r_b - r_a} = \frac{Q_1}{V}$$

$$C_2 = \frac{4\pi K\epsilon_0}{2}\frac{r_a r_b}{r_b - r_a} = \frac{Q_2}{V}$$

$$\Rightarrow \boxed{C_{eq} = 2\pi\epsilon_0 \left(\frac{r_a r_b}{r_b - r_a}\right)(1+K)}$$

b) $\oint \vec{E}\cdot d\vec{a} = \frac{Q_{free}}{K\epsilon_0}$, use a hemispherical Gaussian surface

For the upper: $E\frac{4\pi r^2}{2} = \frac{Q_1}{\epsilon_0} \Rightarrow E = \frac{1}{2\pi\epsilon_0}\frac{Q_1}{r^2}$, $\frac{Q_2}{K} = Q_1$, $Q_1 + Q_2 = Q$

$\Rightarrow Q_1 + KQ_1 = Q$

lower: $E\frac{4\pi r^2}{2} = \frac{Q_2}{K\epsilon_0} \Rightarrow E = \frac{1}{2\pi\epsilon_0}\frac{Q_2}{Kr^2}$

$$\Rightarrow \boxed{E = \frac{1}{2\pi\epsilon_0}\frac{Q}{(1+K)r^2}}$$

same for upper and lower halves

c) $\sigma_{f_u} = \frac{Q_1}{2\pi r_a^2} = \frac{Q}{2\pi r_a^2(1+K)}$, $\sigma_{f_u} = \frac{Q}{2\pi r_b^2(1+K)}$

$\sigma_{f_L} = \frac{Q_2}{2\pi r_a^2} = \frac{KQ}{2\pi r_a^2(1+K)}$, $\sigma_{f_L} = \frac{KQ}{2\pi r_b^2(1+K)}$

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d) Upper plate: $\sigma_{bound\,u} = 0$

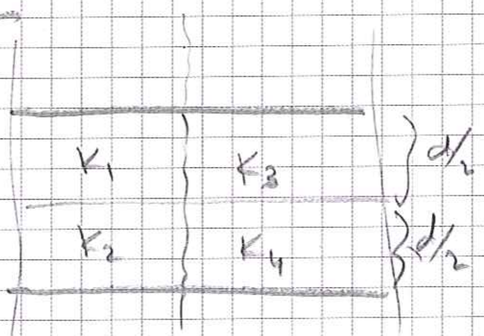
Lower plate: at r_a $\quad \sigma'_{iL} = \sigma'_{freeL}\left(1-\frac{1}{\kappa}\right)$

$$= \frac{\cancel{\kappa} Q}{2\pi r_a^2}\cdot\frac{1}{(1+\kappa)}\cdot\left(\frac{\kappa-1}{\cancel{\kappa}}\right) = \frac{Q}{2\pi r_a^2}\left(\frac{\kappa-1}{\kappa+1}\right)$$

at r_b $\quad \sigma'_{iL} = \frac{Q}{2\pi r_b^2}\left(\frac{\kappa-1}{\kappa+1}\right)$

e) Zero bound charge on the flat surface of the interface.

Ex



$$C_{12} = \frac{2A\epsilon_0}{2d}\left(\frac{K_1K_2}{K_1+K_2}\right)$$

$$C_{34} = \frac{2A}{2}\frac{\epsilon_0}{d}\left(\frac{K_3K_4}{K_1+K_2}\right)$$

$$\rightarrow C_{eq} = C_{12}+C_{34} = \frac{A\epsilon_0}{d}\left(\frac{K_1K_2}{K_1+K_2}+\frac{K_3K_4}{K_3+K_4}\right)$$

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MT1 Review:

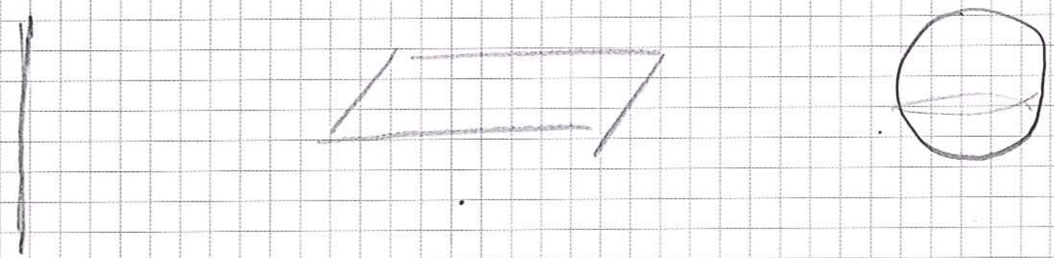
Chap. 21: Electric Charge and Electric Field

$$|\vec{F}| = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

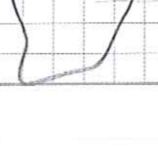
Chap. 22: Gauss' Law

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{tot}}{\epsilon}$$

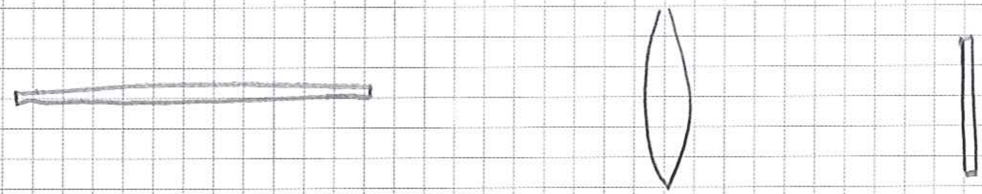


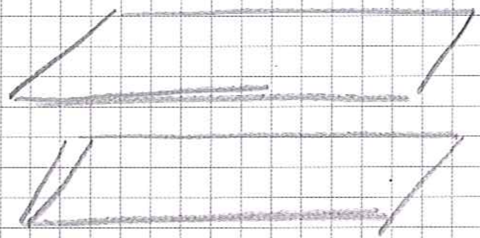
Chap. 23: Electric Potential Energy

$$\int_a^b \vec{F_e}\cdot d\vec{l} = U_a - U_b \quad , \quad V_a - V_b = \int_a^b \vec{E}\cdot d\vec{l}$$



$$V = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r}$$

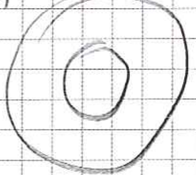




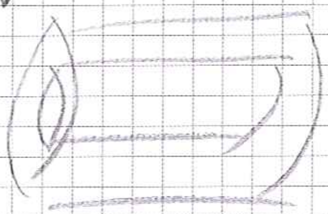
Chap. 24: Capacitance and Dielectrics

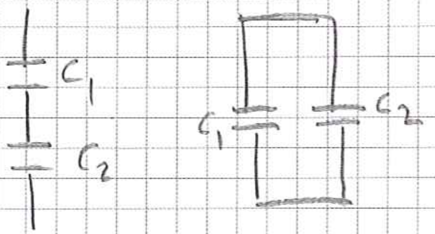
Capacitance calculations: - Assume charges $+Q$ and $-Q$
- Calculate V
- Find $C = \frac{Q}{V}$

Spherical cap



Cylindrical cap





same Q same V

Energy storage: $E = \frac{Q^2}{2C} = \int V\,dq = \int \frac{q}{C}\,dq = \frac{Q^2}{2C}$