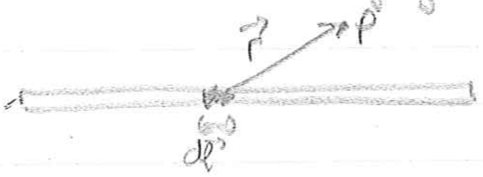
# Chapter 28: Sources of Magnetic Field

How is a Magnetic field generated?

Stationary charges generate Electric Field (Coulomb's Law)

Moving charges (current) generate Magnetic Field (Biot and Savart Law)

Consider a current carrying wire




The magnetic field generated by a piece with length $dl$ is:

$$\boxed{d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{l}\times\hat{r}}{r^2}}$$ Biot and Savart Law

Compare with Coulomb's Law $$\boxed{d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2}\hat{r}}$$

$d\vec{l}$ : vector with length $dl$ in the same direction as the current flow

$\vec{r}$ : vector from the point on the wire to the observation point P. $\hat{r}$ unit vector in the direction of $\vec{r}$. $\hat{r} = \frac{\vec{r}}{r}$

P : observation point

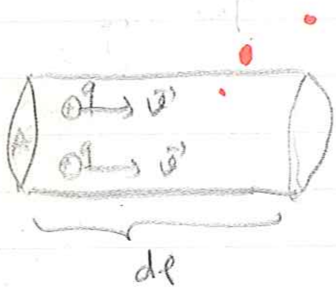
$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0}{4\pi}\int\frac{I\,d\vec{l}\times\hat{r}}{r^2}}$$

integrated over the current carrying wire

$\mu_0$ : magnetic permeability constant

__Examples:__ Magnetic field due to a moving charge q

Consider the individual charges making the current flow




$I = nqvA$

$$\Rightarrow d\vec{B} = \frac{\mu_0}{4\pi}\frac{nqvA\,d\vec{l}\times\hat{r}}{r^2}$$

$\vec{v}$ is in the same direction as current $d\vec{l}$

$$\rightarrow d\vec{B} = \frac{\mu_0}{4\pi}\frac{nq\,dl\,A\;\vec{v}\times\hat{r}}{r^2}$$

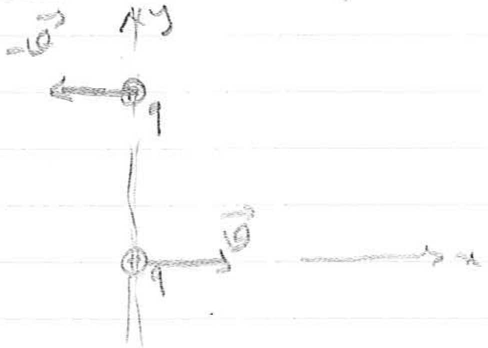
$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Nq\;\vec{v}\times\hat{r}}{r^2}$$

$N$ → number of charges

∴ Field due to a single charge

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2}}$$

Ex 28.1: Two moving charges



Two protons move parallel to x-axis in opposite directions. Electric and magnetic forces on the upper proton and the ratio of their magnitudes?

$$\vec{F}_e = q\vec{E} = q\cdot\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} = \frac{q^2}{4\pi\epsilon_0 r^2}\hat{y}$$

$$\vec{F}_B = q\vec{v}\times\vec{B} = q\left(-v\hat{x}\times\left(\frac{\mu_0}{4\pi}q\frac{\vec{v}\times\hat{r}}{r^2}\right)\right) = -qv\left(\hat{x}\times\left(\frac{\mu_0 q}{4\pi}\frac{v\,\hat{x}\times\hat{y}}{r^2}\right)\right)$$

$$= -\frac{q^2\mu_0 v^2}{4\pi}\left(\frac{\hat{x}\times\hat{k}}{r^2}\right) = -\frac{\mu_0 q^2 v^2}{4\pi r^2}(-\hat{y}) = \frac{\mu_0 q^2 v^2}{4\pi r^2}\hat{y}$$

$$\Rightarrow \frac{F_B}{F_E} = \frac{\mu_0 \cancel{q^2} v^2}{\cancel{4\pi}\cancel{r^2}}\frac{\cancel{4\pi}\epsilon_0\cancel{r^2}}{\cancel{q^2}} = \mu_0\epsilon_0 v^2 = \frac{1}{c^2}v^2 = \boxed{\frac{v^2}{c^2}}$$

∴ when $v \ll c$ $\quad F_B \ll F_e$ !

## Magnetic Field of a Straight Current-Carrying Conductor


$\vec{B}$ at a point a distance $x$ from the conductor on its perpendicular bisector.

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, d\vec{l} \times \hat{r}}{r^2}$$

$$d\vec{l} \times \hat{r} = -dl\hat{k}\sin\phi \Rightarrow d\vec{B} = -\frac{\mu_0}{4\pi} I \frac{dl}{(x^2+y^2)} \hat{k} \left(+\frac{x}{r}\right) \quad , \quad dl = dy$$

$$\Rightarrow B = \int dB = \frac{\mu_0 I}{4\pi} \int_{-a}^{a} \frac{dy\, x}{r^3}$$

$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$ , $\frac{y}{x} = \tan\theta$ $\Rightarrow dy = \frac{x}{\cos^2\theta} d\theta$

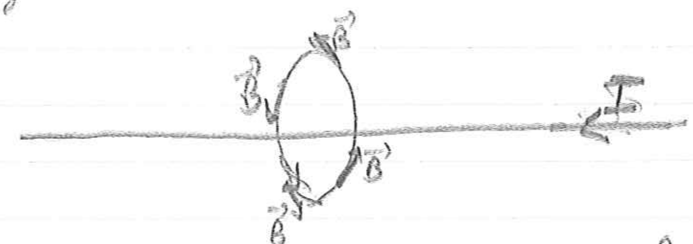
$$= \frac{\mu_0 I}{4\pi} \int \frac{x}{\cos^2\theta} d\theta \cdot \frac{x}{x^3} \cos^3\theta = \frac{\mu_0 I}{4\pi x} \int_{-\theta_{max}}^{\theta_{max}} \cos\theta\, d\theta$$

$$= \frac{\mu_0 I}{4\pi x} 2\sin\theta_{max} = \frac{\mu_0 I}{2\pi x} \cdot \frac{a}{\sqrt{a^2+x^2}}$$

$$B = \frac{\mu_0 I}{2\pi} \frac{a}{x\sqrt{a^2+x^2}}$$

for $a \to \infty$ : $\lim_{a\to\infty} B = \frac{\mu_0 I}{2\pi x}$

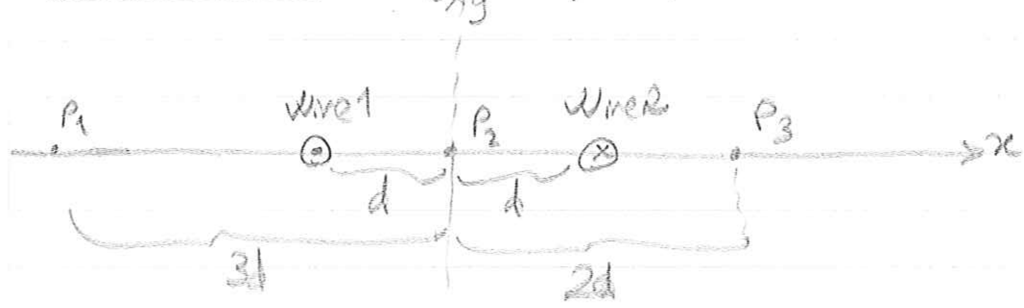
Problem is symmetric for rotations around y-axis. $\vec{B}$ has the same magnitude at all points on a circle centered on the conductor. $\vec{B}$ is in the tangential direction to this circle.




$B = \frac{\mu_0 I}{2\pi r}$ , long straight, current-carrying conductors.

Right hand rule gives the direction.

Example 28.4: Magnetic field of two wires




Each wire carries a current I.

a) Magnitude and direction of $\vec{B}$ at $P_1, P_2, P_3$?

APPLY SUPERPOSITION

at $P_1$ : $\vec{B}_1 = -\frac{\mu_0 I}{2\pi \times 2d}\hat{j} + \frac{\mu_0 I}{2\pi \times 4d}\hat{j} = -\frac{\mu_0 I}{2\pi \times 4d}\hat{j}$ //

$$\vec{B}_2 = \frac{\mu_0 I}{2\pi \times 2d}\hat{j} + \frac{\mu_0 I}{2\pi \times 2d}\hat{j} = \frac{\mu_0 I}{2\pi d}\hat{j} \quad //$$

$$\vec{B}_3 = \frac{\mu_0 I}{2\pi \times 3d}\hat{j} - \frac{\mu_0 I}{2\pi \times d}\hat{j} = -\frac{2\mu_0 I}{2\pi \times 3d}\hat{j} \quad //$$

b) Magnitude and direction of $\vec{B}$ at any points to the right of wire 2?

$$\vec{B}(x) = -\frac{\mu_0 I}{2\pi (x-d)}\hat{j} + \frac{\mu_0 I}{2\pi (x+d)}\hat{j}$$

$$= -\frac{\mu_0 I}{2\pi}\hat{j}\left\{\frac{1}{x-d} - \frac{1}{x+d}\right\} = -\frac{\mu_0 I d}{\pi (x^2-d^2)}\hat{j}$$

for $x \to \infty$ $\boxed{\lim_{x \to \infty} \vec{B}(x) = -\frac{\mu_0 I d}{\pi x^2}\hat{j}}$ //

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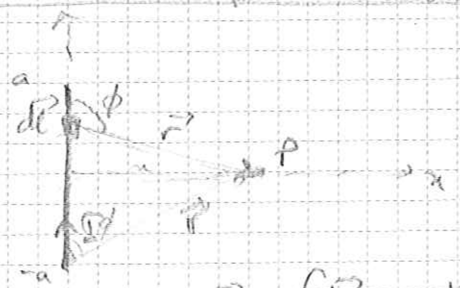
**Summary:** $d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, d\vec{l} \times \hat{r}}{r^2}$ Biot and Savart Law


## Magnetic Field of a Straight Current carrying wire:




$d\vec{l} = dl\hat{j}$ , $\vec{r} = r\sin\phi\hat{i} + r\cos\phi\hat{j}$

$\Rightarrow d\vec{l} \times \hat{r} = -\sin\phi\, \hat{k}\, dl$

$$\Rightarrow \vec{B} = \int d\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{-\sin\phi\, dl\, \hat{k}}{r^2}$$

$$= -\frac{\mu_0 I}{4\pi} \int \frac{x}{r^3} dy\, \hat{k} = -\frac{\mu_0 I}{4\pi} \int_{-a}^{a} \frac{x}{(x^2+y^2)^{3/2}} dy$$

$$= -\frac{\mu_0 I}{2\pi} \frac{a}{x\sqrt{a^2+x^2}} \hat{k}$$

$\lim_{a\to\infty} \vec{B} = -\frac{\mu_0 I}{2\pi x} \hat{k}$ , given by the right hand rule.

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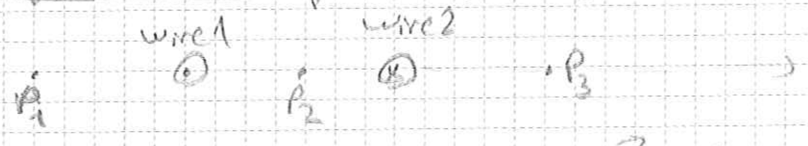
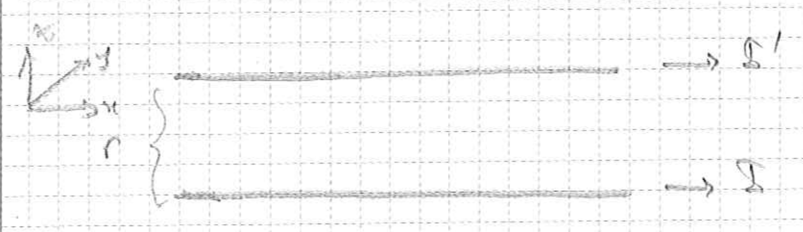
## Magnetic Field of two wires:




Vector addition of $\vec{B}$ due to each wire.

## Force between Parallel Conductors:




$\vec{B} = \frac{\mu_0 I}{2\pi r} (-\hat{j})$

$\Rightarrow$ Force on the upper conductor:

$\vec{F} = I' \vec{L} \times \vec{B}$

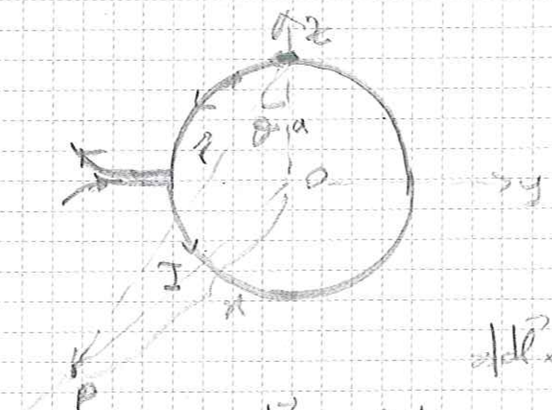
$\vec{F} = I' L \hat{i} \times \frac{\mu_0 I}{2\pi r}(-\hat{j}) = \left(\frac{\mu_0 I I'}{2\pi r} L\right) - \hat{k}$

$\Rightarrow$ force per unit length: $\boxed{\frac{\vec{F}}{L} = -\frac{\mu_0 I I'}{2\pi r}\hat{k}}$ attractive!

For the other wire:

$\frac{\vec{F}_2}{L} = \frac{\mu_0 I I'}{2\pi r}\hat{k}$ : attractive, due to symmetry

Magnetic Field of a Circular Current Loop:



Magnetic field at a point on the axis of the loop.

$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times \hat{r}}{r^2}$

$d\vec{l}$ is orthogonal to $\hat{r}$

$|d\vec{l} \times \hat{r}| = dl$

$$\left.\begin{array}{l} d\vec{l} = -\hat{j}\, dl \\ \hat{r} = -\cos\theta\, \hat{k} + \sin\theta\, \hat{i} \end{array}\right\} \Rightarrow d\vec{l} \times \hat{r} = \cos\theta\, dl\, \hat{i} + \sin\theta\, dl\, \hat{k}$$

$$B_x = \int dB_x = \int \frac{\mu_0 I}{4\pi} \frac{\cos\theta\, dl}{r^2} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{a}{(a^2+x^2)^{3/2}}\, a\, d\phi = \frac{\mu_0 I a^2}{2(a^2+x^2)^{3/2}}$$

$B_z = 0$ due to rotational symmetry

$B_x = \frac{\mu_0 \mu}{2\pi (x^2+a^2)^{3/2}}$ , $\mu = I\pi a^2 \Leftarrow$ magnetic dipole moment

$$\vec{dl} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$
$$\vec{dl} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}$$
$$\vec{dl} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$
$$\vec{dl} = d\rho\,\hat{\rho} + \rho\,d\theta\,\hat{\theta} + dz\,\hat{z}$$



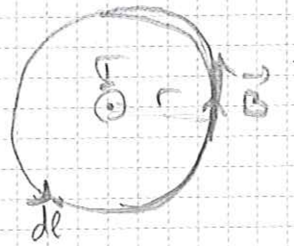
## Ampère's Law

Similar to the Gauss' Law $\oint \vec{E}\cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0}$, there is a simple relationship between $\vec{B}$ and $I$.

The relationship now involves:

$\oint \vec{B}\cdot\vec{dl}$ ; line integral of $\vec{B}$ around a closed path.

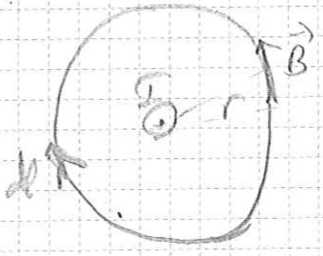
Consider the magnetic field caused by a long, straight, conductor carrying a current $I$:




$B = \frac{\mu_0 I}{2\pi r}$, in the tangential direction.

$$\Rightarrow \oint \vec{B}\cdot\vec{dl} = \int B\hat{\phi}\cdot dl\,\hat{\phi} = B\int dl = 2\pi r B = \mu_0 I$$

circle with radius r
integration path
counter clockwise



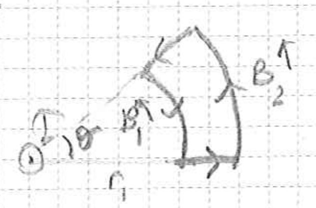

$$\oint \vec{B}\cdot\vec{dl} = \int B\hat{\phi}\cdot dl(-\hat{\phi})$$

clockwise direction

$$= -B\int dl = -B2\pi r = -\mu_0 I$$

$\Rightarrow \underbrace{\oint \vec{B}\cdot\vec{dl}}$ $= \mu_0 I$ — current which obeys the right-hand rule.

closed path integral around a circle.

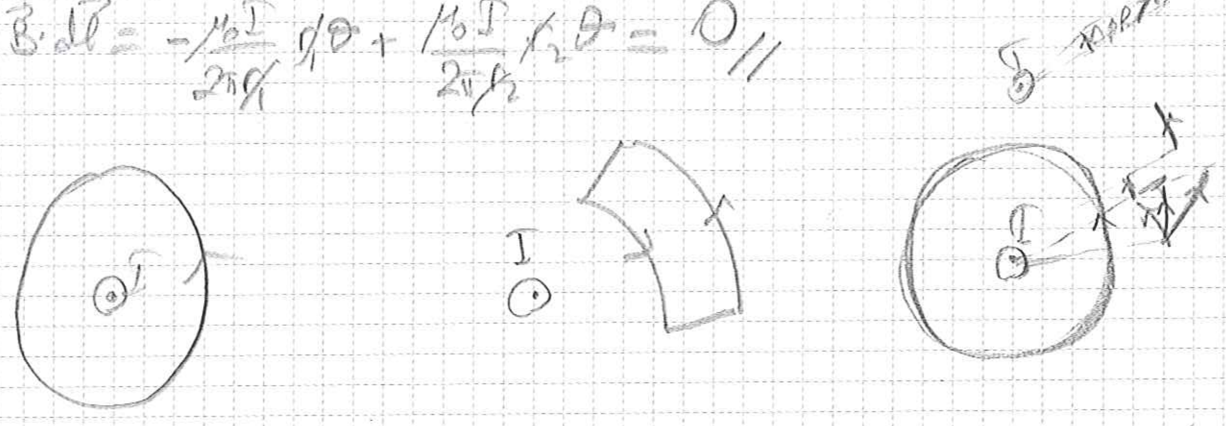



Now consider a closed path that does not enclose the current.

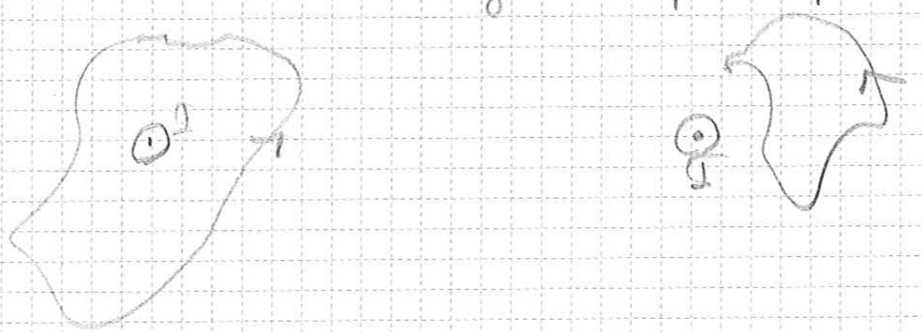
$$\oint \vec{B}\cdot\vec{dl} = \int_{①} -B_1\,dl + \int_{②} B_2\,dl$$

$$= -B_1\int_{①} dl + B_2\int_{②} dl = -B_1\frac{2\pi r_1\theta}{2\pi} + B_2\frac{2\pi r_2\theta}{2\pi}$$

$$\Rightarrow \oint \vec{B}\cdot d\vec{l} = -\frac{\mu_0 I}{2\pi r_1} r_1 \theta + \frac{\mu_0 I}{2\pi r_2} r_2 \theta = 0$$




These results considering the special paths can be generalized.




$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I \qquad \oint \vec{B}\cdot d\vec{l} = 0 = \mu_0 \underset{\substack{\uparrow \\ 0}}{I}$$

enclosed current.

If there are more than one current carrying wires:


Superposition:

$$\mu_0 I_1 + \mu_0 I_2 + \ldots = \oint \vec{B}_1\cdot d\vec{l} + \oint \vec{B}_2\cdot d\vec{l} + \ldots$$

$$\Rightarrow \mu_0 I_{encl} = \oint \vec{B}\cdot d\vec{l}$$

$$I_{encl} = I_1 + I_2 + \ldots$$

$$\vec{B} = \vec{B}_1 + \vec{B}_2 + \ldots$$

$\therefore$ Ampère's Law:

$$\boxed{\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}}$$

$\rightarrow$ algebraic sum of currents enclosed by the integration path.

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## Applications of Ampère's Law:

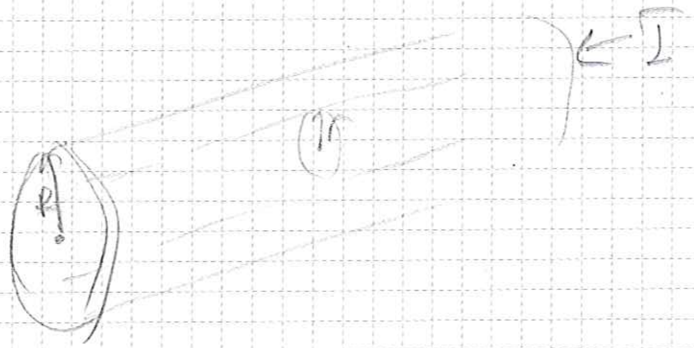
We will benefit from the symmetry of the problems and use the Ampère's Law to determine the $\vec{B}$ field.

Ex: 28.8.
Field of a long, straight, current-carrying wire:


Due to symmetry $\vec{B}$ with a distance r from the wire should have the same magnitude and tangential direction.

$$\Rightarrow B 2\pi r = \mu_0 I \Rightarrow B = \frac{\mu_0 I}{2\pi r}$$

Ex 28.9: Field inside a long cylindrical conductor



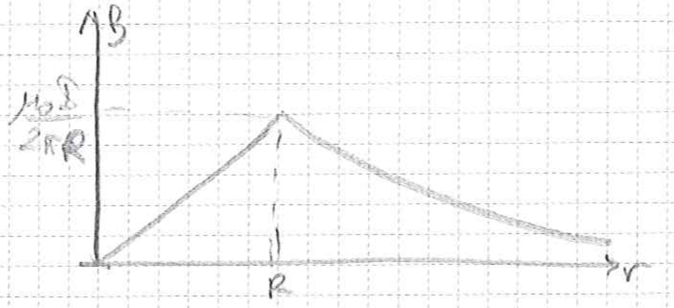

Consider a long cylindrical conductor with a radius R, carrying a current I.

The field at a distance r<R from the center:

$$\oint \vec{B}\cdot d\vec{l} = B 2\pi r = \mu_0 \frac{I}{\pi R^2}\pi r^2 \Rightarrow B = \frac{\mu_0 I}{2\pi}\frac{r}{R^2} = \frac{\mu_0 I}{2\pi R^2} r$$

At a distance r>R from the center

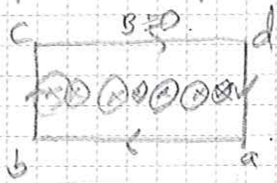
$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I = B 2\pi r \Rightarrow B = \frac{\mu_0 I}{2\pi r}$$




Ex. 28.12: Field of a solenoid:


Field lines towards the center of a solenoid are nearly parallel. External fields are almost equal to zero.

$\Rightarrow$ Choose an amperian loop:




$$\Rightarrow \oint \vec{B} \cdot d\vec{l} = BL = \mu_0 I N$$

number of loops enclosed

if $n = \frac{N}{L}$, number of turns per unit length

$$\Rightarrow \boxed{B = \mu_0 I n}$$

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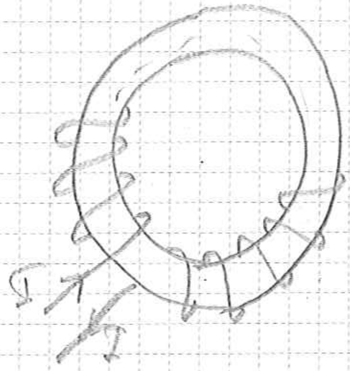
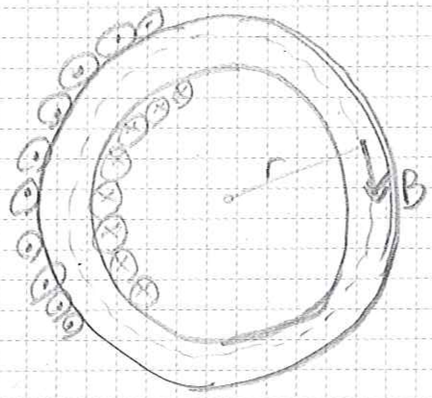
Ex. 28.11:

Field of a toroidal solenoid.




Consider a toroidal solenoid having a total of N turns.




If the coils are very tightly wound, we can consider circular loops in the solenoid.

$\Rightarrow$ B is constant and tangential in direction.

Consider a circular amperian loop:

$$B 2\pi r = \mu_0 I N \Rightarrow \boxed{B = \frac{\mu_0}{2\pi r} I N}$$ in the tangential direction.

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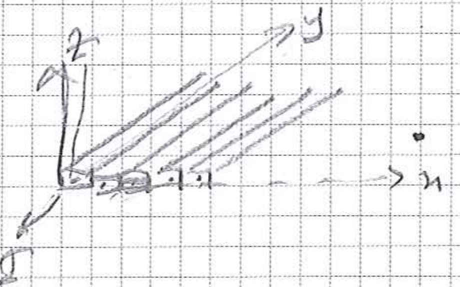
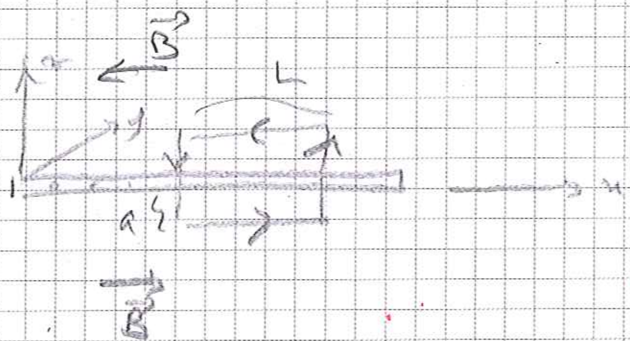
Prob 28.79: Infinite current sheet.




Long, straight conductors with square cross-sections each carrying current I are laid side-by-side to form an infinite current sheet.

There are n conductors per unit length along the x-axis.

Magnitude and direction of the magnetic field at a distance a below and above the current sheet?




Due to symmetry

$$\oint \vec{B}\cdot d\vec{l} = 2BL = \mu_0 I n L \Rightarrow \boxed{B = \mu_0 \frac{In}{2}}$$ constant

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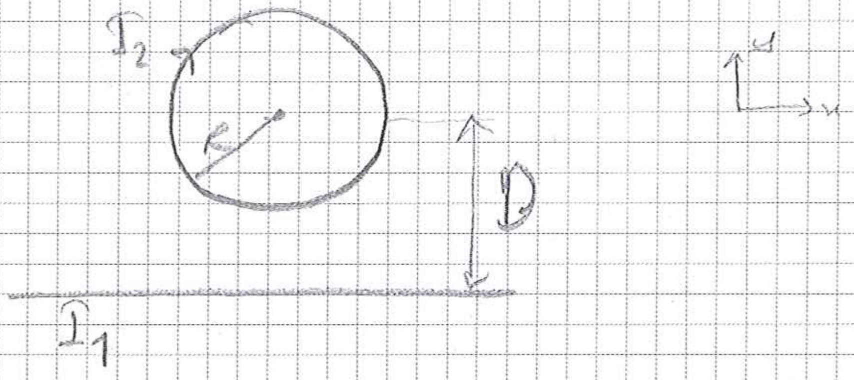
Prob 28.24:




What are the magnitude and direction of $I_1$ if $\vec{B}=0$ at the center of the circular loop?

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\,\frac{d\vec{l}\times\hat{r}}{r^2} \Rightarrow \vec{B}_2 = \frac{\mu_0 I_2}{4\pi}\cdot\frac{1}{R^2}\,2\pi R(-\hat{k})$$

$$= \frac{\mu_0 I_2}{2}\,\frac{-\hat{k}}{R}$$

$I_1$ should flow in $+x$ direction.

$$\Rightarrow \vec{B}_1 = \frac{\mu_0 I_1}{2\pi D}\,\hat{k}$$

$$B_1 = B_2 \Rightarrow \frac{\cancel{\mu_0} I_2}{\cancel{2}}\,\frac{1}{R} = \frac{\cancel{\mu_0} I_1}{\cancel{2}\pi D} \Rightarrow \boxed{I_1 = \frac{\pi D}{R}\,I_2}$$

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Prob: 28.68

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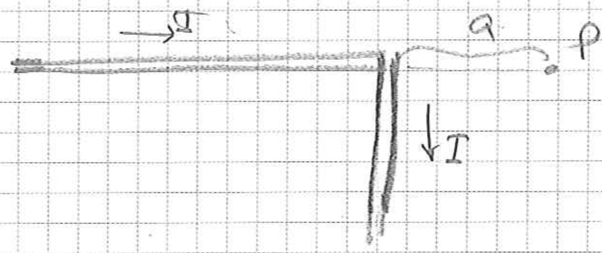
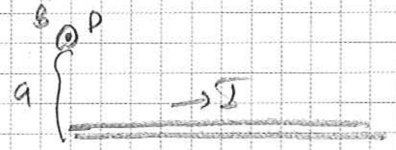
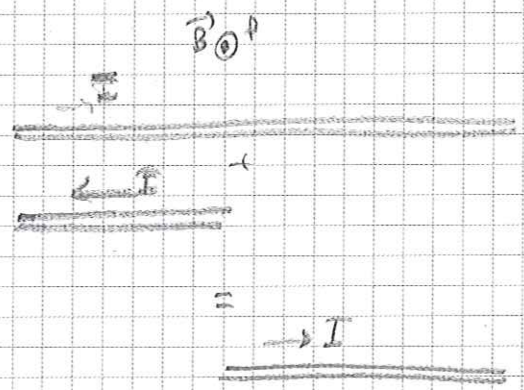



The wire is infinitely long and carries a current I. What is $\vec{B}$ at point P?


$\vec{B} = 0$




$\vec{B} = ?$




$$B = \frac{\mu_0 I}{2\pi a} - B \Rightarrow \boxed{B = \frac{\mu_0 I}{4\pi a}}$$