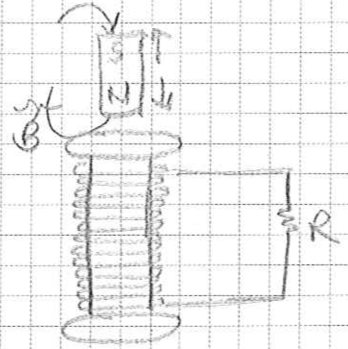
Chapter 29: Electromagnetic Induction



Consider a coil and a magnet.

Experimental observation;

- When the magnet is at rest there is no current flowing in the coil.
- When the magnet is moved toward or away from the coil, current starts flowing in the coil.

⇒ Change in the magnetic field generates current. This is called the phenomenon of electromagnetic induction.

Faraday's Law

Electromagnetic induction is governed by the Faraday's Law. Consider a closed loop:

Faraday's law states that a change in the magnetic flux flowing through the loop will generate an emf given as:

$$\boxed{\mathcal{E} = -\frac{d\Phi_B}{dt}}, \quad \Phi_B = \int \vec{B}\cdot d\vec{A}$$

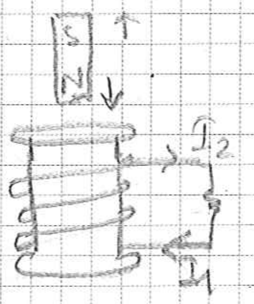
inside a source of emf $V_{ab} = \mathcal{E}$

Induced electric field direction

$\mathcal{E}$ is the induced emf.

given by the Lenz's Law

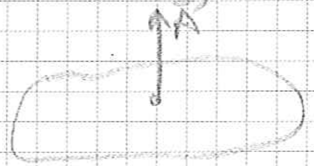
Induced emf ($\mathcal{E}$) has a specific direction. The direction is given by the Lenz's Law ⇒ Induction effect will generate a magnetic field which is opposite to the change in the total magnetic field.



- If the magnet is moved toward the coil ⇒ the coil will try to decrease the B field ⇒ flow direction is I_1
- If the magnet is moved away from the coil ⇒ flow direction is I_2

⇒ Direction of $\mathcal{E}$ is the same as the direction of I.

Formally, to determine the direction of the induced emf or the current:



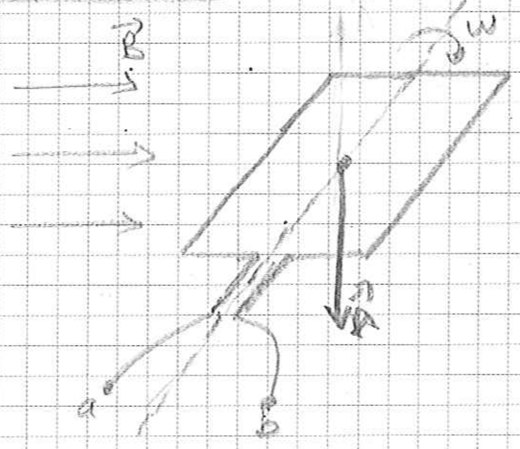
- Define a positive direction for the loop
- Determine the sign of induced emf.

$\left(-\frac{d\phi_B}{dt}\right) \rightarrow$ if $\frac{d\phi_B}{dt} > 0 \Rightarrow$ sign of $\mathcal{E}$ is negative

if $\frac{d\phi_B}{dt} < 0 \Rightarrow$ sign of $\mathcal{E}$ is positive

- Apply the right hand rule.
  - If the sign of induced emf is + $\Rightarrow$ your thumb should point in the direction of $\vec{A}$
  - If the sign of induced emf is − $\Rightarrow$ your thumb should point in the direction opposite to $\vec{A}$.

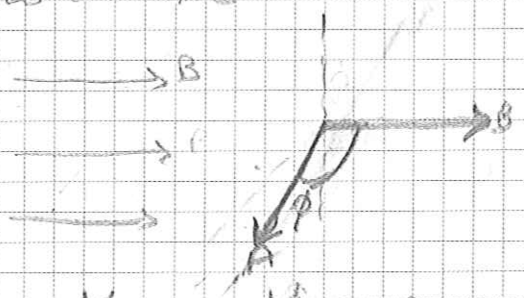
Example 29.4:



A rectangular loop is made to rotate with constant angular frequency w. Magnetic field $\vec{B}$ is uniform.
$\Rightarrow$ Determine the induced emf.

use $\mathcal{E} = -\frac{d\Phi_B}{dt}$

- Choose the direction of the vector $\vec{A}$
- At a certain time t:



$\Rightarrow \Phi_B = \int \vec{B} \cdot d\vec{A}$

$\Phi_B = BA\cos\phi = BA\cos wt$

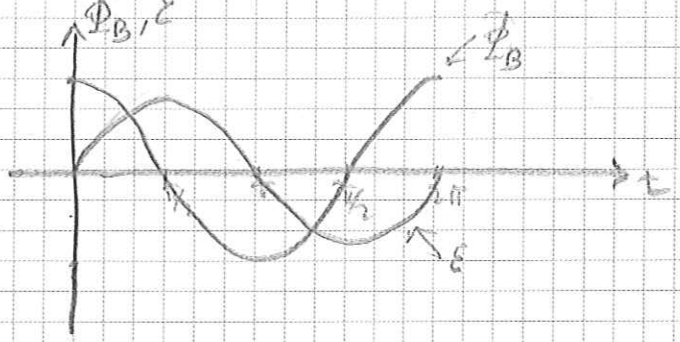
and $w = \frac{d\phi}{dt} \Rightarrow \frac{d\Phi_B}{dt} = -BA\sin\phi \frac{d\phi}{dt} = -BA\,w\sin\phi$

$\Rightarrow$ Induced emf: $\mathcal{E} = -\frac{d\Phi_B}{dt} = BAw\sin\phi = BAw\sin wt$

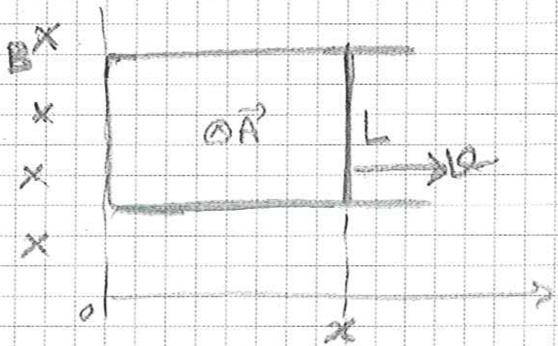
$\phi = wt$

At $\phi = 0$ Φ_B is maximum

for increasing ϕ Φ_B decreases $\rightarrow$ direction of induced current is from a to b



Example 29.6:



Consider a metal rod with length L across the two arms of a U-shaped conductor.

The rod is moved towards right at a speed v.

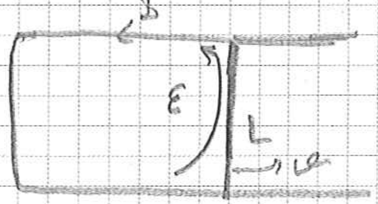
Magnitude and direction of the resulting emf.

Consider $\vec{A}$ to be in the direction $\otimes$

$\Rightarrow \Phi_B$ at time t : $\Phi_B(t) = BA(t) = BLx(t)$

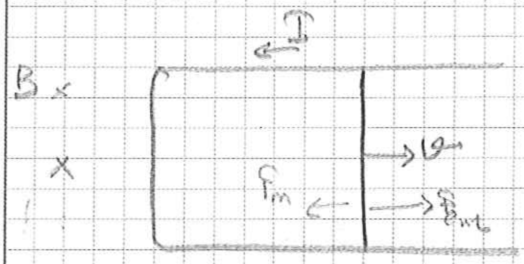
$$\Rightarrow \mathcal{E} = -\frac{d\Phi_B}{dt} = -BL\frac{dx}{dt} = -BLv$$ //

The direction of $\mathcal{E}$:



$\mathcal{E}$ Try to decrease the total flux.

Ex 29.7: Work and power in the slidewire generator



Work done by the force pulling the rod should be equal to the rate of energy dissipated in the circuit.

F_{ext} : force pulling the rod
F_m : magnetic force
since v is constant
$F_{ext} = F_m$, no accelaration.

Work done by F_{ext}:

$$\frac{d}{dt}(F_{ext}\Delta x) = F_{ext}\frac{dx}{dt} = F_{ext}v = ILBv \quad //$$

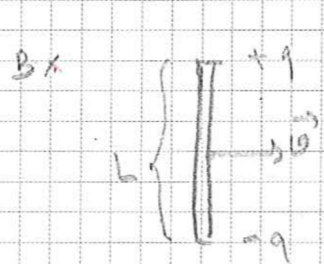
Rate of energy dissipated in the circuit:

$$\mathcal{E}I = BLvI \quad //$$

∴ Mechanical energy is converted into electrical energy.

29.4 Motional Electromotive Force

Consider a single conducting rod moving with a constant velocity under a constant magnetic field.



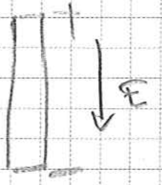
A charge element q will be subject to a magnetic force

$$\vec{F}_m = q\vec{v}\times\vec{B}$$

⇒ Positive charges will be collected on one side and negative charges on the other side.

At equilibrium there will be no net charge flow in the conducting rod.

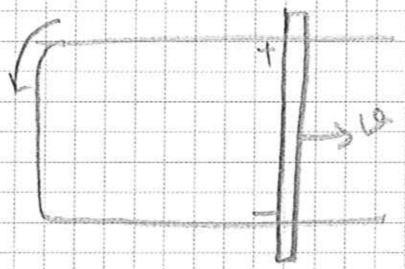
$\vec{F}_e = q\vec{E} = q\vec{v}\times\vec{B} \Rightarrow \cancel{q}E = \cancel{q}vB \Rightarrow \boxed{E = vB}$



Potential difference on the rod

$V_{+-} = EL = vBL$

$\therefore \boxed{\mathcal{E} = vBL}$ is called the motional emf.



$\mathcal{E} = vBL$ motional emf

or Faraday's Law

$\mathcal{E} = -\frac{d\phi}{dt} = -vBL$

They explain the same physical phenomenon.

In problems with moving conductors, motional emf can be applied rather than the Faraday's Law.

For a non straight conductor

Assume a portion of the conductor with length $d\vec{l}$, On this portion;

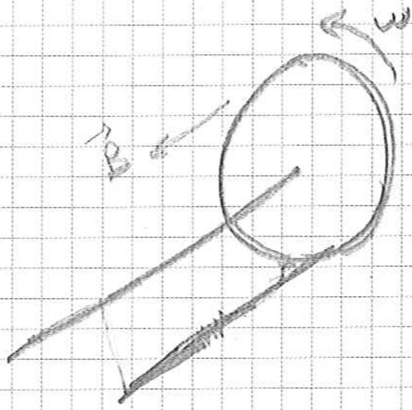
$d\mathcal{E} = \vec{E}\cdot d\vec{l} \Rightarrow \cancel{q}\vec{E} = q\vec{v}\times\vec{B}$

$\Rightarrow d\mathcal{E} = (\vec{v}\times\vec{B})\cdot d\vec{l}$

$\Rightarrow$ On the whole conductor $\boxed{\mathcal{E} = \int d\mathcal{E} = \int(\vec{v}\times\vec{B})\cdot d\vec{l}}$

"Homopolar generator"

Example 29.11: Faraday Disk dynamo.



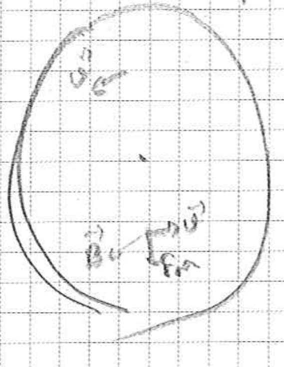
Consider a conducting disk with radius R, rotating with angular velocity ω about its axis going through the center in the normal direction.

The disk is subject to a constant Magnetic field $\vec{B}$.

Induced emf between the center and the rim of the disk?

Consider a charge q on the disk. This charge will move with velocity $\vec{v}$ in the tangential direction.

⇒ Magnetic force $\vec{F}_m = q\vec{v}\times\vec{B}$ is in the radial direction!



⇒ At equilibrium a positive potential difference will built up between the rim and the center. Such that charges flowing in the circuit do not accelerate. In the radial direction:

$$\vec{F}_m = q\vec{v}\times\vec{B} = q\vec{E} \Rightarrow \vec{E} = \vec{v}\times\vec{B}$$

$$E(r) = B\underset{\substack{\uparrow \\ \text{speed} \\ \text{at different} \\ \text{radial location}}}{v(r)} = B\omega r$$

~~$I =$ const. $J(r) = \frac{I(r)}{2\pi r}$~~

~~$J = \sigma E$~~

~~$\Rightarrow E(r) = \frac{I}{\sigma}\frac{1}{2\pi r}$~~

$$\Rightarrow \mathcal{E} = \int E(r)\,dr = \int_0^R B\omega r\,dr = \frac{1}{2}\omega B R^2$$ // motional emf

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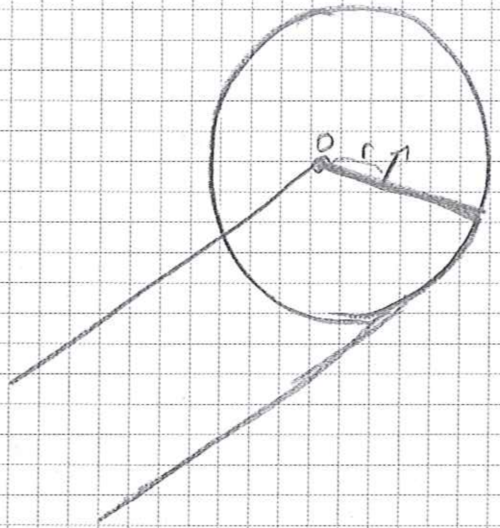

$$\mathcal{E} = \int (\vec{\vartheta} \times \vec{B}) \cdot d\vec{l}$$

$$\vec{\vartheta}(r) = r w \hat{\varphi} \ , \ \vec{B} = B \hat{z} \ , \ d\vec{l} = dr\, \hat{a}_r$$

$$\Rightarrow \vec{\vartheta} \times \vec{B} = r w B \hat{r}$$

$$\Rightarrow \mathcal{E} = \int_0^R r w B \, dr = \frac{1}{2} w B R^2 \ //$$

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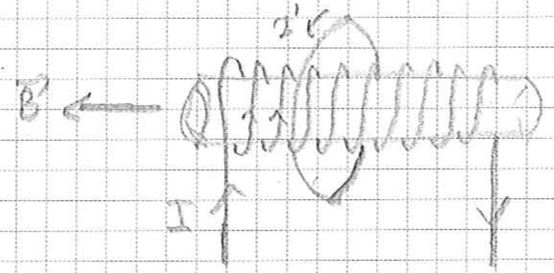
2.5 Induced Electric Fields:

Consider a Solenoid and a wire loop.



If the current in the solenoid changes with time, an emf will be induced in the wire loop.

$B = \mu_0 n I$, (using Ampère's law)

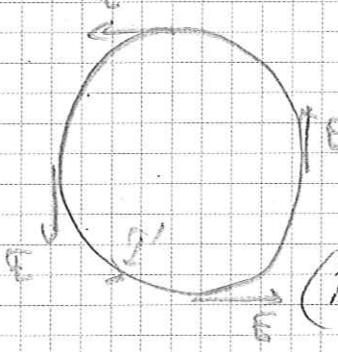
⇒ Flux through the wire loop: $\Phi_B = \int \vec{B}\cdot d\vec{A} = \mu_0 n I A$

⇒ $\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}(\mu_0 n I A) = -\mu_0 n A \frac{dI}{dt}$ //

If the loop has a total resistance R the induced current in the loop is:

$I = \frac{\mu_0 n A}{R}\frac{dI}{dt}$, in magnitude.

Direction is given by Lenz's law,



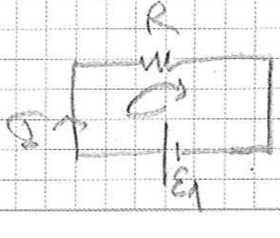
$$\Rightarrow \boxed{\mathcal{E} = \oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt}}$$

Even if the conductor is not moving in a magn. field, induced electric field can be obtained.

(Stationary integration path) induced electric field.

∴ When a charge q goes once around the loop the total work done on it is $q\mathcal{E}$. ⇒ The electric field in the loop is not conservative.

Normally:



$\oint \vec{E}\cdot d\vec{l} = 0 \Rightarrow \mathcal{E}_1 - IR = 0$

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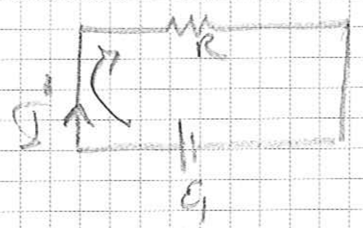
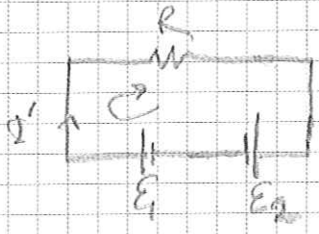
In the case of induction

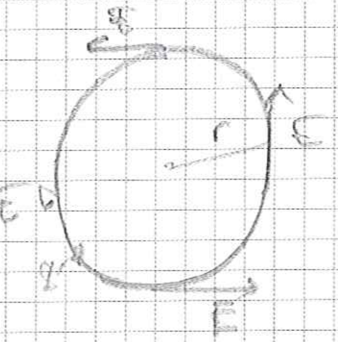


$$\oint \vec{E}\cdot d\vec{l} = \mathcal{E}_1 - IR = -\frac{d\Phi_B}{dt} = \underbrace{\frac{\mu_0 n A}{R}\frac{dI}{dt}}_{\mathcal{E}_2}$$

Effectively



Consider the simple situation of a circular wire loop.



$$\oint \vec{E}\cdot d\vec{l} = E 2\pi r = \left|\frac{d\Phi_B}{dt}\right|$$

$$\Rightarrow \boxed{E = \frac{1}{2\pi r}\left|\frac{d\Phi_B}{dt}\right|} \quad //$$

29.7 Displacement Current and Maxwell's Equations:

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\left(\int \vec{B}\cdot d\vec{A}\right)$$ ⇐ Varying magnetic field generates electric field.

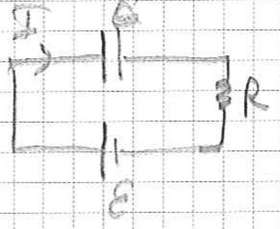
⇒ Would varying electric field generate magnetic field?

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}$$, Ampère's Law

In this statement magnetic field generation is not governed by the electric field.

Indeed Ampère's Law is incomplete!
Varying electric field also generates magnetic field.

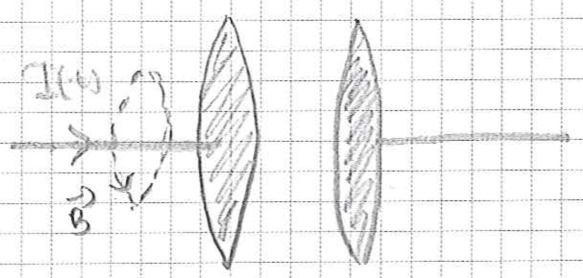
This is most obvious in the case of charging of a capacitor.



Consider this circuit and the capacitor is initially uncharged.

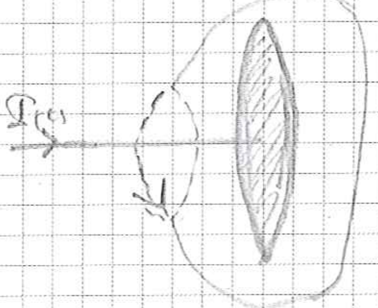
At a time t:

Parallel plate capacitor.



$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl} = \mu_0 I(t)$ for the plane surface area.

Now consider the same loop but a surface encompassing the first plate of the capacitor.



$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} = 0$$

But we know

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I(t)$$

∴ There is a missing term.

Considering uniform electric field between the parallel plates:

$$I(t) = \frac{dQ}{dt} = \frac{d}{dt}(CV) = C\frac{d}{dt}(Ed) = Cd\frac{dE}{dt}$$

$$= \epsilon_0 \frac{A}{d} d \frac{dE}{dt} = \epsilon_0 A \frac{dE}{dt} = \epsilon_0 \frac{d\Phi_E}{dt} = I(t)$$

∴ If we write $\oint \vec{B} \cdot d\vec{l} = \underset{\substack{\swarrow \\ \text{conduction} \\ \text{current}}}{\mu_0 I_{(t)}} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$, the Ampère's Law is valid for both areas.

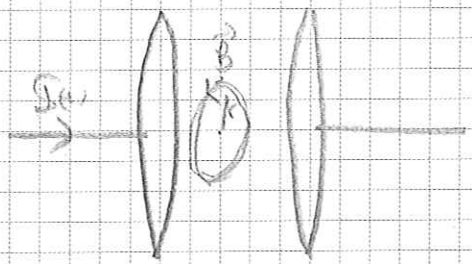
This equation is indeed valid for all situations:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_C(t) + \mu_0 I_D$$

$$\swarrow$$

$$I_D = \epsilon_0 \frac{d\Phi_E}{dt} = \epsilon_0 \frac{d}{dt} \int \vec{E} \cdot d\vec{A} \text{ ; is called } \sout{\text{the}} \text{ displacement current}$$

Consider the measurement of B between the parallel plates:



If the plates are circular, $\vec{B}$ at a distance r from the center is constant in magnitude and tangential in direction.

$$\Rightarrow \underbrace{\oint \vec{B} \cdot d\vec{l}}_{} = \mu_0 \underbrace{I_{enc}}_{0} + \mu_0 \frac{d}{dt} \int \vec{E} \cdot d\vec{A}$$

$$B 2\pi r = 0 + \mu_0 \frac{d}{dt} \int \vec{E} \cdot d\vec{A} = \mu_0 A \frac{d}{dt} E$$

if the loop covers the whole area;

$$I_D = I_{enc} = I(t)$$

$$\therefore \quad I_D(r) = \frac{I(t)}{A(R)} A(r) = \frac{I(t)}{\pi R^2} \pi r^2 = \frac{I(t) r^2}{R^2}$$

$$\Rightarrow B = \frac{1}{2\pi r} \mu_0 \frac{I(t) r^2}{R^2} = \boxed{\frac{\mu_0 r}{2\pi R^2} I(t)}$$

//

Maxwell's Equations of Electromagnetism

Now we can write down all the relationships relating the electric field with the magnetic field:

Modified Ampère's Law;
$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \left(I_c + \epsilon_0 \frac{d}{dt}\int \vec{E}\cdot d\vec{A} \right)$$

Faraday's Law:
$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{A}$$

Gauss' Law: for electric field
$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0}$$

Gauss' Law for magnetic field:
$$\oint \vec{B}\cdot d\vec{A} = 0$$
, no magnetic monopoles

All the basic relations between fields and their sources are contained in Maxwell's equations.

Maxwell's Equations together with the Equation of the Lorentz Force
$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$
constitute all the fundamental relations of Electromagnetism.

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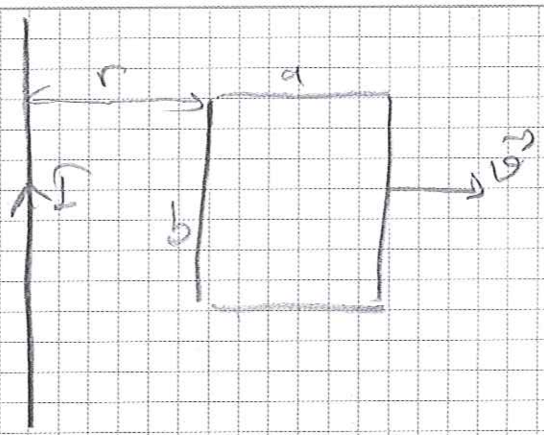


The Loop is being pulled to the right at constant speed v. Magnitude and direction of the ~~magnetic~~ induced emf.

$$\Phi_B = \int \vec{B}\cdot d\vec{A} = \int \frac{\mu_0 I}{2\pi r}\, dr\, dy = \frac{\mu_0 I}{2\pi} L \int_r^{r+a} \frac{dx}{x}$$

$$= \frac{\mu_0 I L}{2\pi} \ln\left(\frac{r+a}{r}\right)$$

$$\Rightarrow \frac{d\Phi_B}{dt} = \frac{\mu_0 I L}{2\pi}\left(\frac{r}{r+a}\right)\left(\frac{r-(r+a)}{r^2}\right)\frac{dr}{dt}$$

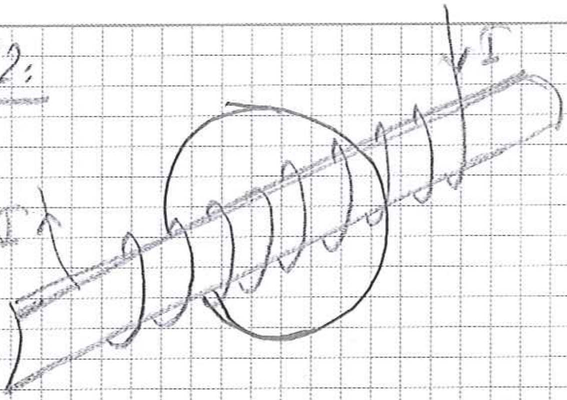
$$= \frac{\mu_0 I L}{2\pi}\,\frac{\cancel{r}}{r+a}\,\frac{-a}{r^{\cancel{2}}}\,\frac{dr}{dt} = -\frac{\mu_0 I L}{2\pi}\,\frac{a}{r(r+a)}\, v \quad //$$

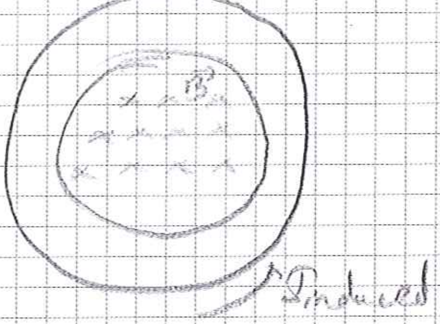
$$\Rightarrow \mathcal{E} = -\frac{d\Phi_B}{dt} = \frac{\mu_0 I L}{2\pi}\,\frac{a}{r(r+a)}\, v \quad //$$

Direction → clockwise.

112.2

Ex 29.12:





Suppose a long solenoid is wound with 500 turns/meter, and the current in the windings is increasing at the rate of 100 A/s. Cross-sectional area of the solenoid is $4 \times 10^{-4} m^2$.

a) Magnitude of the induced emf in the wire loop outside the solenoid

b) Magnitude of the induced electric field, if the wire loop has a radius of 2cm

a) $\mathcal{E} = -\frac{d\Phi_B}{dt}$, $\Phi_B = BA = \mu_0 I n A$

$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$ $BL = \mu_0 I n L \Rightarrow B = \mu_0 I n$

$\Rightarrow \mathcal{E} = -\mu_0 n A \frac{d\Phi_B}{dt} = -4\pi \times 10^{-7} \times 500 \times 4 \times 10^{-4} \times 100$

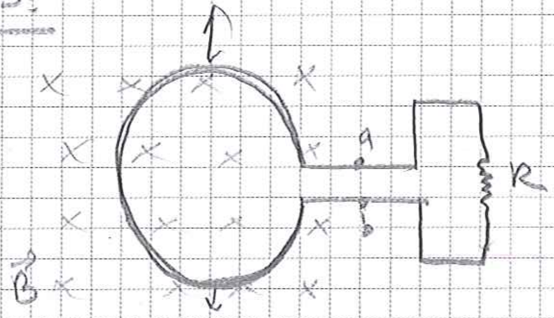
$= -25 \mu V$ //

b) $\mathcal{E} = \oint \vec{E} \cdot d\vec{l} = 2\pi r E$, due to symmetry

$\Rightarrow E = \frac{\mathcal{E}}{2\pi r} = \frac{25 \times 10^{-6}}{2\pi \times 2 \times 10^{-2}} = 2 \times 10^{-4} V/m$ //

112.3

Prob 29. 53:



A flexible circular loop of diameter 6.5cm lies in a B-field of 0.95 T.

The loop is pulled forming a loop of zero area in 0.25s.

a) Average induced emf in the circuit?

b) Direction of current in R?

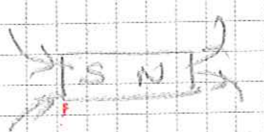
a) $\mathcal{E} = -\frac{d\Phi_B}{dt} \Rightarrow \mathcal{E}_{avg} = -\frac{\Delta\Phi_B}{\Delta t} = -\frac{\left(0 - (0.95)\pi \times \frac{(6.5\times10^{-2})^2}{4}\right)}{0.25}$

$= 0.0126\ V$

b) From a to b.

Review:

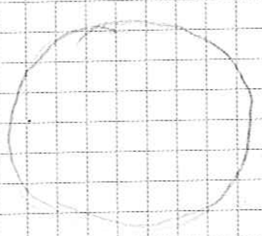
Chapter 27: Magnetic Field and Magnetic Forces



Moving charges (current) generate magnetic field.

Magnetic Force on a moving charge $\vec{F}_M = q(\vec{v} \times \vec{B})$

- $\vec{F}_M$ always perpendicular to the direction of motion
  $\Rightarrow \vec{F}_M$ does no work
- In a medium with constant magnetic field



$$|\vec{F}_M| = qvB = \frac{mv^2}{R} \Rightarrow R = \frac{mv}{qB}$$

cyclotron frequency:

$$\frac{v}{f} = 2\pi \frac{mv}{qB} \Rightarrow f = \frac{qB}{2\pi m}$$

$$T = 2\pi \frac{m}{qB}$$

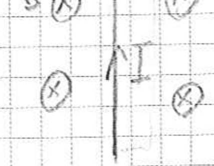
Helical path if the velocity has a component parallel to B.

- In a medium where there is both electric and magnetic fields:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \text{ ; Lorentz Force}$$

Magnetic Force on a Current-Carrying Conductor:

$$\int d\vec{F} = \int dq \, \vec{v} \times \vec{B}$$

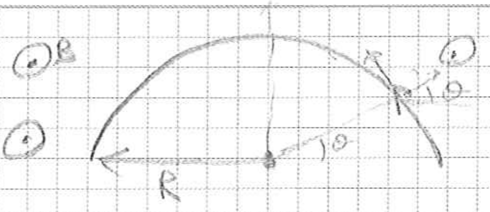


$$I = \frac{v \Delta t}{\Delta t} n A q = q n A v$$

$$\Rightarrow \int d\vec{F} = q n A L \vec{v} \times \vec{B} = I \vec{L} \times \vec{B} \Rightarrow \boxed{\vec{F}_M = I \vec{L} \times \vec{B}}$$

vector along the wire

$$\boxed{d\vec{F} = I \, d\vec{l} \times \vec{B}}$$



$\vec{F}_x = 0$

$dF_y = I R d\theta B \sin\theta$

$\Rightarrow F_y = \int_0^{\pi} IRB \sin\theta \, d\theta = IRB\,2 = 2IRB$ //

- Force and Torque on a current Loop:

Net Force = 0

Net Torque:

$\tau_1 = F_1 \frac{b}{2} \sin\phi$, $F_1 = IaB$

$\tau_2 = F_2 \frac{b}{2} \sin\phi$, $F_2 = IaB$

$\Rightarrow \tau = IabB\sin\phi$

$= IAB\sin\phi = |\vec{\mu} \times \vec{B}| \Rightarrow \vec{\mu} = I\vec{A}$

Work done by F_1, F_2:

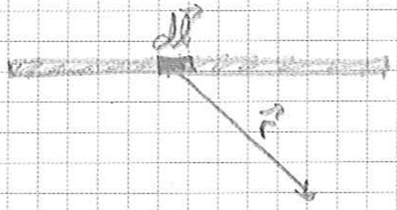
$dW = \tau_1 d\phi + \tau_2 d\phi = -IAB\sin\phi\, d\phi$

$\Rightarrow W = \int_{\phi_1}^{\phi_2} -IAB\sin\phi\, d\phi = +IAB(\cos\phi_2 - \cos\phi_1) = -\Delta U = U_1 - U_2$

$\Rightarrow U = -IAB\cos\phi = \boxed{-\vec{\mu}\cdot\vec{B}}$ //

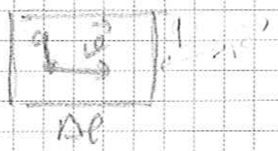
$\vec{\tau} = \vec{\mu} \times \vec{B}$, $\vec{\mu} = I\vec{A}$

Chapter 28: Sources of Magnetic Field



$$\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \hat{r}}{r^2}$$, Biot and Savart Law.

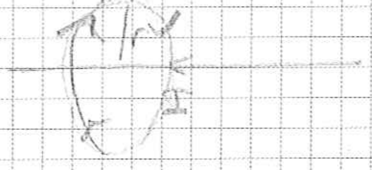
Consider a single moving charge



$$I\Delta\vec{l} = \frac{\Delta Q}{\Delta t}\Delta\vec{l} = q\frac{\Delta\vec{l}}{\Delta t} = q\vec{v}$$

$$\Rightarrow \vec{B} = \frac{\mu_0}{4\pi} q \frac{\vec{v} \times \hat{r}}{r^2} \quad //$$

Magnetic Field of a Straight Current-Carrying Wire:

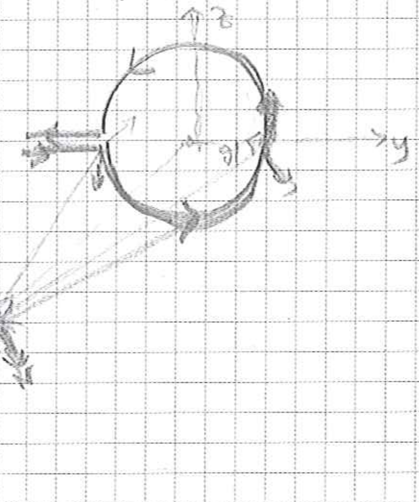


$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$$

$\boxed{\text{Ampere's Law}}$

$$\Rightarrow B = \frac{\mu_0 I}{2\pi r}$$

Magnetic Field of a Circular Current Loop:

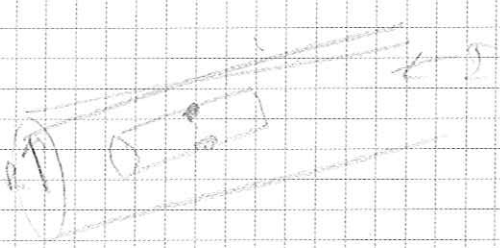


$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times \hat{r}}{r^2}$$

$$\int dB_x = \int \frac{\mu_0 I}{4\pi} \frac{dl \cos\theta}{r^2} = \frac{\mu_0 I}{4\pi} \frac{\cos\theta}{r^2} \int dl$$

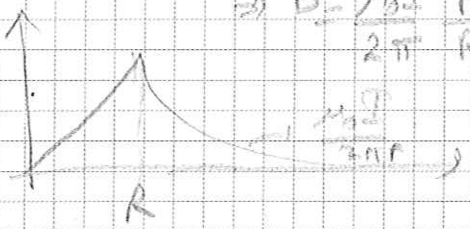
$$= \frac{\mu_0 I}{4\pi} \frac{\cos\theta}{r^2} 2\pi a$$

$$= \frac{\mu_0 I}{\cancel{4}\pi} \frac{\cancel{2}\pi a^2}{r^3} = \frac{\mu_0 I}{2} \frac{a^2}{(x^2+a^2)^{3/2}} \quad //$$



$$\oint \vec{B}\cdot d\vec{l} = B\,2\pi r = \frac{\mu_0 I}{\pi R^2}\pi r^2$$

$$\Rightarrow B = \frac{\mu_0 I}{2\pi}\frac{r}{R^2}$$



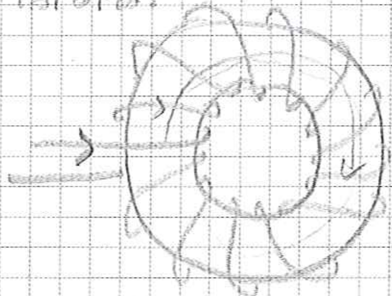
Solenoid

Uniform B

$$BL = \mu_0 I N$$

$$\Rightarrow B = \mu_0 I n \,,\; n = \frac{N}{L}$$

Toroid:

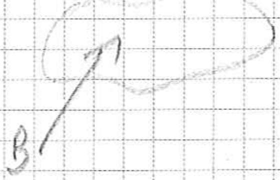


Constant B at radius r

$$B\,2\pi r = \mu_0 I N \Rightarrow B = \frac{\mu_0 I N}{2\pi r}$$

Chapter 29: Electromagnetic Induction

Faraday's Law:



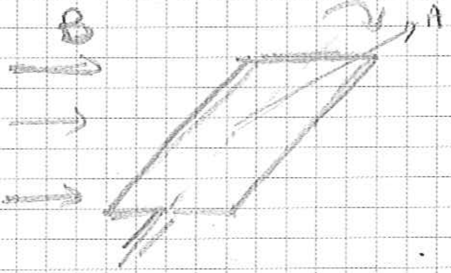
$$\mathcal{E} = -\frac{d\Phi_B}{dt} \quad , \quad \Phi_B = \int \vec{B}\cdot d\vec{A}$$

induced emf

direction is given by Lenz's Law

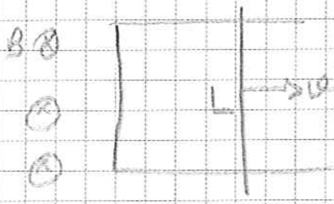
Increasing B $\Rightarrow$ I_1

Decreasing B $\Rightarrow$ I_2



$$\Phi_B = BA\cos\phi = BA\cos\omega t$$

$$\Rightarrow \mathcal{E} = -\frac{d\phi_B}{dt} = BA\omega\sin\omega t \,//$$



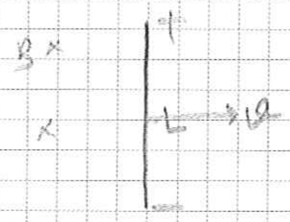
$$\Phi_B = BLx \Rightarrow \mathcal{E} = -\frac{d\Phi_B}{dt} = -BLv \,//$$

energy converted to electrical energy

$$F = BIL \ , \ F\frac{dx}{dt} = BILv \,//$$

energy dissipated in the circuit $\mathcal{E}I = BLvI \,//$

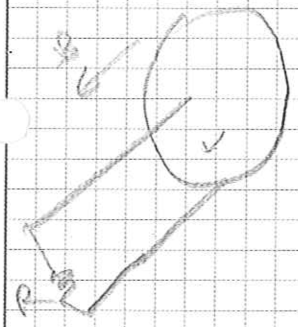
Motional emf:



At equilibrium

$$qvB = qE \Rightarrow E = Bv$$

$$\mathcal{E} = \int E dl = EL = BvL \; //$$



Disk dynamo,

$$qvB = qE \Rightarrow E = B\omega r$$

$$\int E dr = \int B\omega r dr = B\omega \frac{R^2}{2} = V_{+-} = \mathcal{E}$$

Induced Electric Fields:

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = \underset{\text{induced } E}{\oint \vec{E} \cdot d\vec{l}}$$

Maxwell's Eq.s;

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \left(I + \epsilon_0 \frac{d}{dt} \int \vec{E} \cdot d\vec{A} \right)$$

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0}$$

$$\oint \vec{B} \cdot d\vec{A} = 0$$

+ Lorentz Force $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$