Math 324 Spring 2013
Midterm-II

Name: ______________________________ Student No : ______________

Signature: ___________________________ Section : ______________

1	10	
2	10	
3	10	
4	10	
5	10	
Total	**50**	

- Complete all questions for full credit.
- You may use a scientific calculator and one hand-written 8.5 by 11 inch page of notes (two sided). Write your name on your notesheet and turn it in with your exam.
- You may use a scientific calculator. Graphic calculators (and any other electronic devices) are not allowed.
- If you don't show your work clearly then you will not receive full credit. Correct answer without a work (such as guessing or trial and error) may not get any point.
- You have 50 minutes to complete the exam.
- Any student found in academic misconduct will receive 0 on this exam (and will be reported).

1. Let $F=(P,Q,R)$ where $P=x+2y-z,\quad Q=xy^2z,\ R=P^4+P^2Q$.

(a) Let $u=\left(\frac{3}{5},0,\frac{4}{5}\right)$ Calculate $D_u\,[D_u\,Q]$

$$\begin{aligned} g &= D_uQ \\ &= u\cdot\nabla Q \\ &= u\cdot(y^2z, Q_y, xy^2) \\ &= \frac{3}{5}y^2z+\frac{4}{5}xy^2 \end{aligned}$$

$$\begin{aligned} D_ug &= u\cdot\nabla g \\ &= u\cdot\left(\frac{4}{5}y^2, g_y, \frac{3}{5}y^2\right) \\ &= \frac{12}{25}y^2+\frac{12}{25}y^2 \\ &= \frac{24}{25}y^2 \end{aligned}$$

Note: no need to calculate Q_y and g_y.

Answer =

(b) Calculate div F at $(1,1,1)$. (Hint : Chain rule)

$P=x+2y-z,\quad Q=xy^2z,\ R=P^4+P^2Q$

$\operatorname{div} F(1,1,1)=\frac{\partial P}{\partial x}(1,1,1)+\frac{\partial Q}{\partial y}(1,1,1)+\frac{\partial R}{\partial z}(1,1,1)$

$$\frac{\partial P}{\partial x}(1,1,1) = 1 \qquad \frac{\partial Q}{\partial y}(1,1,1)=2$$

$$\begin{aligned} \frac{\partial R}{\partial z}(x,y,z) &= \frac{\partial R}{\partial P}\frac{\partial P}{\partial z}+\frac{\partial R}{\partial Q}\frac{\partial Q}{\partial z}=(-4P^3-2PQ)+(P^2xy^2) \\ \frac{\partial R}{\partial z}(1,1,1) &= -42^3-2\cdot 2+2^2=-32 \\ & \text{since } P(1,1,1)=2 \text{ and } Q(1,1,1)=1 \end{aligned}$$

Thus $\operatorname{div} F(1,1,1)=1+2-32=-29$

Answer =

2. Let C consists of 2 line segments from $(0,0)$ to $(10,1)$ then from $(10,1)$ to $(\pi,0)$.

(a) Evaluate the line integral

$$\int_C (x+y)\,\mathrm{d}x + x^2\,\mathrm{d}y,$$

C_1: $r(t) = (10t, t), \quad 0 \leqslant t \leqslant 1,$ $\qquad$ C_2: $r(t) = (10 - 6.86t, 1 - t), \quad 0 \leqslant t \leqslant 1$

$$\begin{aligned}
\int_{C_1} (x+y)\,\mathrm{d}x + x^2\,\mathrm{d}y &= \int_0^1 110t + 100t^2\,\mathrm{d}t = \left(55t^2 + \frac{100}{3}t^3\right)\Big|_0^1 = 55 + \frac{100}{3} \\
\int_{C_2} (x+y)\,\mathrm{d}x + x^2\,\mathrm{d}y &= \int_0^1 -6.86(11 - 7.86t) - (10 - 6.86t)^2\,\mathrm{d}t \\
&= \int_0^1 -47.04t^2 + 191.12t - 175.46\,\mathrm{d}t \\
&= -\frac{1}{3}47.04 + \frac{1}{2}191.12 - 175.46 \\
&= -95.59
\end{aligned}$$

Answer = $55 + \frac{100}{3} - 95.59 \approxeq -7.25$

(b) If $f(x,y) = \sin(x/6)\cos(y)$ then evaluate the line integral

$$\int_C \nabla f \cdot \mathrm{d}r$$

From fundamental theorem of line integrals

$$\begin{aligned}
\int_C \nabla f \cdot \mathrm{d}r &= f(\pi, 0) - f(0,0) \\
&= \sin(\pi/6)\cos(0) - \sin(0)\cos(0) \\
&= 1/2
\end{aligned}$$

Answer =

3. Let $F = (P, Q)$ where

$$P = \ln y + 14xy^3, \quad Q = 21x^2y^2 + \frac{x}{y}$$

Is it possible to find a function f such that $F = \nabla f$? Explain.

If your answer is YES then calculate f,

If your answer is NO then calculate $\frac{\partial^2 P}{\partial x y} - \frac{\partial^2 Q}{\partial x^2}$

The domain of F is $\{(x, y) \mid y > 0\}$ which is open and simply connected. Moreover

$$\frac{\partial P}{\partial y} = \frac{1}{y} + 42xy^2 = \frac{\partial Q}{\partial x},$$

thus F is conservative so YES there exists a function f such that $F = \nabla f$.

$$f_x = \ln y + 14xy^3 \quad \Longrightarrow \quad f(x, y) = x \ln y + 7x^2y^3 + g(y)$$

$$f_y = 21x^2y^2 + \frac{x}{y} \quad \Longrightarrow \quad f(x, y) = x \ln y + 7x^2y^3 + h(x)$$

Thus

$$f(x, y) = x \ln y + 7x^2y^3 + C$$

Answer =

4. Use Green's Theorem to evaluate the following line integral along the given positively oriented curve.

$$\int_C 3y^3\,\mathrm{d}x - 3x^3\,\mathrm{d}y$$

where C is the circle $x^2 + y^2 = 4$.

$$\begin{aligned}
\int_C 3y^3\,\mathrm{d}x - 3x^3\,\mathrm{d}y &= \int\int_D -9x^2 - 9y^2\,\mathrm{d}A \\
&= -9\int_0^{2\pi}\int_0^2 r^2\,r\,\mathrm{d}r\,\mathrm{d}\theta \\
& \quad -18\pi\int_0^2 r^3\mathrm{d}r \\
&= -18\pi\,\frac{1}{4}r^4|_0^2 \\
&= -72\pi
\end{aligned}$$

Answer =

5. Solve each one separately

(a) Surface S is given by $r(u,\theta)=(u\,,u\theta,u\sin\theta),\ 0\leqslant u\leqslant 1,\ 0\leqslant\theta\leqslant\pi$. Setup (**Donot Evaluate**) and integral that calculates the area of S. (The integral should be in terms of u and θ not r)

$$\begin{aligned} r_u &= (1,\theta,\sin\theta) \\ r_\theta &= (0,u,u\cos\theta) \\ |r_u\times r_\theta| &= |(u\theta\cos\theta-u\sin\theta,-u\cos\theta,u\,)| \\ &= \sqrt{(u\theta\cos\theta-u\sin\theta)^2+u^2\cos^2\theta+u^2} \end{aligned}$$

$$\text{Area}(S)=\int_S \text{dS}=\int_0^1\int_0^\pi \sqrt{(u\theta\cos\theta-u\sin\theta)^2+u^2\cos^2\theta+u^2}\,\text{d}\theta\text{d}u$$

Answer =

(b) For each of the below choose one : T for true, F for false

- $\nabla\cdot\nabla\times\nabla\times F=0$ T
- If F is conservative then $\nabla\cdot F=\nabla\times F$ F
- $\nabla\times\nabla(\nabla\cdot F)=0$ T
- If $\nabla\times F\neq 0$ then F is not conservative T
- Every constant vector field F is conservative T
- There is a vector field G such that $\nabla\times G=(x,y,z)$ F