Math 324, Spring 2013. **Quiz #1**

Problem 1 (10 Pts). Find the area of the surface that satisfies $x^2+y^2+z^2=10^2$ and $x\geqslant 5$.

Area $=$ 100π

i. By symmetry we can work on $x^2+y^2+z^2=10^2$ and $z\geqslant 5$. Thus we want to find the surface area of $z=\sqrt{100-(x^2+y^2)}$ that lies above the disk $D=\{(x,y)\,|\,x^2+y^2\leqslant 75\}$. (the intersection of the plane and inside the sphere)

ii. From surface area formula we get

$$\int_D \sqrt{\frac{x^2}{100-(x^2+y^2)}+\frac{y^2}{100-(x^2+y^2)}+1}\ \mathrm{d}A = \int_D \frac{10}{\sqrt{100-(x^2+y^2)}}\ \mathrm{d}A$$

iii. Convert into polar coordinates

$$\int_D \frac{10}{\sqrt{100-(x^2+y^2)}}\ \mathrm{d}A = \int_0^{2\pi}\int_0^{\sqrt{75}} \frac{10r}{\sqrt{100-r^2}}\ \mathrm{d}r\,\mathrm{d}\theta$$

iv. Calculate the integral

$$\int_0^{2\pi}\int_0^{\sqrt{75}} \frac{10r}{\sqrt{100-r^2}}\ \mathrm{d}r\,\mathrm{d}\theta = -2\pi 2\cdot 5(100-r^2)^{1/2}\Big|_0^{\sqrt{75}} = 100\pi$$

Problem 2 (10 Pts). Our aim is to calculate the following integral in 3 steps.

$$\int_0^1\int_x^1 \sqrt{y^2+2}\ \mathrm{d}y\,\mathrm{d}x$$

(a) Find the domain D of integration.

$D=\{(x,y)\ \ \vert\, 0\leqslant x<1,\quad x\leqslant y\leqslant 1\}$

(b) Change the order of integration.

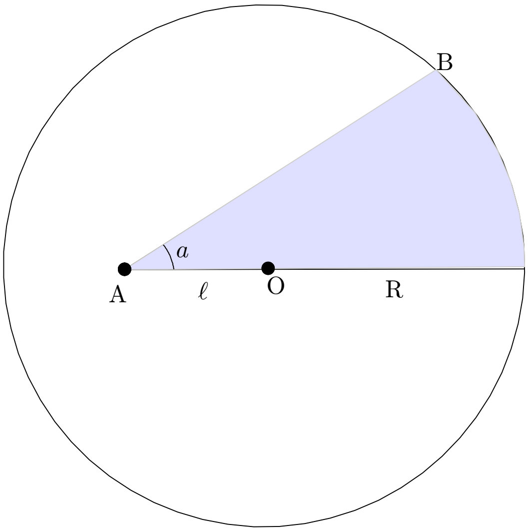
$\int_0^1\int_0^y \sqrt{y^2+2}\ \mathrm{d}x\mathrm{d}y$

(c) Evaluate the integral you find in previous part.

Answer$=$ $\frac{1}{3}(3^{3/2}-2^{3/2})=\sqrt{3}-\frac{2\sqrt{2}}{3}$

$$\begin{aligned}\int_0^1\int_0^y \sqrt{y^2+2}\mathrm{d}x\mathrm{d}y &= \int_0^1 y\sqrt{y^2+2}\ \mathrm{d}y\\ &= \frac{1}{3}(y^2+2)^{3/2}\Big|_0^1 = \frac{1}{3}(3^{3/2}-2^{3/2})\end{aligned}$$

Bonus Question. (5 Pts) Set up an integral that calculates the shaded area as a function of radius R, angle a, and $\ell = |\mathrm{AO}|$. For simplicity assume that $0 < \ell < R$ and $0 < a < \pi/4$.



$$\text{Area} = \int_0^a \int_0^{\frac{2\ell\cos(\theta) \pm \sqrt{4\ell^2\cos^2(\theta) - 4\left(\ell^2 - R^2\right)}}{2}} r\,\mathrm{d}r\,\mathrm{d}\theta$$

i. Impose a polar coordinate system and set A as the origin, so that area can be calculated by

$$\int_0^a \int_0^{g(\theta)} r\,\mathrm{d}r\,\mathrm{d}\theta$$

where $g(\theta)$ corresponds to the curve which is intersection of shaded region and the circle.

ii. Next we need to find $g(\theta)$ in our new coordinate system. In cartesian coordinates (where A is the origin) the circle is given by : $(x - \ell)^2 + y^2 = R^2$, thus

$$(r\cos(\theta) - \ell)^2 + (r\sin(\theta))^2 = R^2.$$

After simplification

$$r^2 - 2r\ell\cos(\theta) + \ell^2 - R^2 = 0$$

Solving this for r gives two values

$$r_{1,2} = \frac{2\ell\cos(\theta) \pm \sqrt{4\ell^2\cos^2(\theta) - 4(\ell^2 - R^2)}}{2}$$

iii. Finally we need to choose the greater solution, since $\theta < \pi/4$.