(Q1) Solve each part separately.

(Q1a, 3pts) Determine which of the differential operators are Linear. Select L for linear, N for nonlinear. Derivatives are with respect to x and $y = y(x)$.

[L/N] $\quad L[y] = 3x^2y + 5xy'$

[L/N] $\quad L[y] = y^2 + y'$

~~[L/N]~~ $\quad \cancel{L[y] = 3y'''' - 2'}$

(Q1b, 3pts) Show that if $L_1[y]$ and $L_2[y]$ are linear differential operators then

$$L[y] = L_1[y] + L_2[y]$$

is also a linear differential operator.

$$L[\alpha y_1 + \beta y_2] = L_1[\alpha y_1 + \beta y_2] + L_2[\alpha y_1 + \beta y_2]$$

$$= \alpha L_1[y_1] + \beta L_1[y_2] + \alpha L_2[y_1] + \beta L_2[y_2]$$

$$= \alpha\underbrace{\left[L_1[y_1] + L_2[y_1]\right]}_{} + \beta\underbrace{\left[L_1[y_2] + L_2[y_2]\right]}_{}$$

$$= \alpha L[y] + \beta L[y_2]$$

(Q1c, 4pts) Suppose that $y_1(x), y_2(x)$ and $y_3(x)$ are linearly independent solutions of

$$L[y] = 0, \qquad y(0) = 1$$

and $y_4(x)$ is a solution of

$$L[y] = g(x), \qquad y(0) = 2.$$

Write two linearly independents solutions $y_5(x), y_6(x)$ of

$$L[y] = g(x), \qquad y(0) = 0.$$

$y_5(x) =$ $y_4 - y_1 - y_2$

$y_6(x) =$ $y_4 - 2y_3$

others: $y_4 - y_1 - y_3$, ... , etc. $y_4 - \frac{2}{3}(y_1 + y_2 + y_3)$, ...

(Q2a, 5pts) Suppose that L is a linear differential operator with ~~constant and~~ real coefficients, and

$$y_1(x) = x + \cos^2(x) + i\sin(x^2)$$

is a solution of

$$L[y] = e^{ix}.$$

Find a solution to

$$L[y] = \cos(x).$$

$$\mathrm{Re}\left[L[y_1]\right] = \mathrm{Re}\left[e^{ix}\right]$$

L has real coefficients

$$L\left[\mathrm{Re}[y_1]\right] = \cos(x)$$

Thus $\mathrm{Re}[y_1] = x + \cos^2(x)$ is a solution.

(Q2b, 5pts) If

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

is a solution to

$$y''' + y = e^x, \qquad y(0) = 2,\ y'(0) = 1,\ y''(0) = 0$$

calculate a_5.

T.S. of e^x

$y(0) = 2 \Rightarrow a_0 = 2$

$y'(0) = 1 \Rightarrow a_1 = 1$

$y''(0) = 0 \Rightarrow a_2 = 0$

$$y''' + y = e^x$$

$$\sum_{n=3}^{\infty} n(n-1)(n-2)\, a_n x^{n-3} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Setting coeffs. of equal powers:

$$3\cdot 2\cdot 1\, a_3 + a_0 = 1$$

$$4\cdot 3\cdot 2\cdot a_4 + a_1 = 1$$

$$5\cdot 4\cdot 3\, a_5 + a_2 = \frac{1}{2}$$

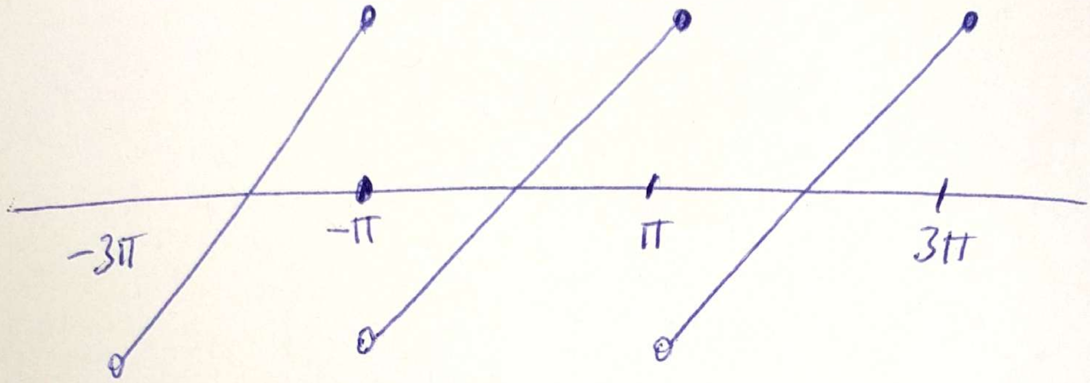
$$\vdots$$

Thus $a_5 = \dfrac{1}{2\cdot 3\cdot 4\cdot 5} = \dfrac{1}{120}$ since $a_2 = 0$

(Q3, 8pts) Work out the Fourier series of 2π periodic function

$$f(x) = x, \quad x \in (-\pi, \pi].$$

At which values of x, if any, does the series fail to converge to $f(x)$? To what values does it converge at those points? (Calculate the integrals).



2π periodic so $l = \pi$

$$\text{FS } f = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x \, dx = 0 \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx = 0$$

odd functions integrated over symmetric domain.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = \frac{1}{\pi}\left[\frac{-x\cos(nx)}{n}\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{-\cos(nx)}{n}dx$$

$$= \frac{-\cos(n\pi)}{n} + \frac{\cos(-n\pi)}{n} = \frac{-2\cos(n\pi)}{n} = \frac{-2(-1)^n}{n}$$

Thus $\boxed{\text{FS } f = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(nx)}$

FS f fails to converge $f(x)$ at discontinuities of f:

$$\text{FS } f(x) = 0 \qquad x \in \{\ldots, -\pi, \pi, 3\pi, \ldots\}$$

(Q4, 10pts) Find a particular solution $y_p(x)$ of

$$y'' + \alpha y = e^x, \qquad \alpha \in \mathbb{R}.$$

(Your answer should depend on α.)

Homogeneous soln: $\quad y'' + \alpha y = 0 \qquad \lambda^2 + \alpha = 0$

$\alpha = 0 \Rightarrow y_h = ax + b$

$\alpha < 0 \Rightarrow y_h = a e^{\sqrt{-\alpha}x} + b e^{-\sqrt{-\alpha}x}$

$\alpha > 0 \Rightarrow y_h = a\cos(\sqrt{\alpha}x) + b\sin(\sqrt{\alpha}x)$

y_p: undetermined coeffs.

$\{e^x\} \leftarrow$ there is duplication when $\alpha = -1$

No duplication when $\alpha \neq -1$

Case 1 $\alpha \neq -1$

$y_p = a e^x$

$a e^x + \alpha a e^x = e^x$

$a = (1+\alpha)^{-1}$

Thus

$$\boxed{y_p = (1+\alpha)^{-1} e^x}$$

Case 2 $\alpha = -1$

$y_p = a x e^x$

$2a e^x + \cancel{a x e^x} - \cancel{a x e^x} = e^x$

$a = \frac{1}{2}$

$$\boxed{y_p = \frac{1}{2} x e^x}$$

(Q5, 8pts) Given the following Strum-Liouville problem, solve for eigenvalues and eigenfunctions and work out the eigenfunction expansion of the function $f(x) = 50$. (Calculate all the integrals)

$$y'' + \lambda y = 0$$
$$y'(0) = 0,\ y(L) + y'(L) = 0.$$

Solutions

$\lambda = 0 \Rightarrow y = ax + b$ BC. $y'(0) = 0 = a \Rightarrow a = 0$

$y(L) + y'(L) = 0 \Rightarrow b = 0$

$y = 0 \Rightarrow \lambda = 0$ not an eigenvalue

$\lambda \neq 0$ (by Thm 17.7.2 in fact $\lambda > 0$) $\Rightarrow$ $y = a\cos(\sqrt{\lambda}\, x) + b\sin(\sqrt{\lambda}\, x)$

BC $y'(0) = 0 = 0 + b\sqrt{\lambda} \Rightarrow b = 0$

so $y(x) = a\cos\sqrt{\lambda}\, x$

$y(L) + y'(L) = 0 \Rightarrow a\cos\sqrt{\lambda}L - a\sqrt{\lambda}\sin\sqrt{\lambda}L$

$\Rightarrow \sqrt{\lambda} = \dfrac{\cos(\sqrt{\lambda}L)}{\sin(\sqrt{\lambda}L)}$ denote solutions as λ_n

$y_n = \cos(\sqrt{\lambda}\, x)$, $\lambda_1 < \lambda_2 < \dots$ are eigenvalues.

$$f(x) = \sum_{n=1}^{\infty} \frac{\langle f, y_n\rangle}{\langle y_n, y_n\rangle} y_n(x)$$

$$\langle f, y_n\rangle = \int_0^L 50\cos(\sqrt{\lambda_n}x)\,dx = \frac{50\sin\sqrt{\lambda_n}L}{\sqrt{\lambda_n}}$$

$$\langle y_n, y_n\rangle = \int_0^L \cos^2(\sqrt{\lambda_n}\,x)\,dx = \int_0^L \frac{1+\cos(2\sqrt{\lambda_n}x)}{2} = \frac{1}{2}\left(L + \frac{\sin(2\sqrt{\lambda_n}L)}{2\sqrt{\lambda_n}}\right)$$

Thus $\boxed{f(x) = \sum_{n=1}^{\infty} \frac{100\sin(\sqrt{\lambda_n}L)}{L\sqrt{\lambda_n} + \sin(2\sqrt{\lambda_n}L)/2}}$

(Q6) Recall the power series (P.S.) solution theorem :

> **THEOREM 4.2.4** *Power series solution*
> If p and q are analytic at x_0, then every solution of
> $$y'' + p(x)y' + q(x)y = 0 \tag{17}$$
> is too, and can therefore be found in the form
> $$y(x) = \sum_0^\infty a_n(x - x_0)^n. \tag{18}$$
> Further, the radius of convergence of every solution (18) is at least as large as the smaller of the radii of convergence of TS $p|_{x_0}$ and TS $q|_{x_0}$.

(Q6a, 7pts) Using power series solve

$$y'' = xy, \qquad y(0) = 7, \quad y'(0) = 9.$$

$$y'' = xy$$

$$2a_2 + 3\cdot 2\, a_3 x + 4\cdot 3\, a_4 x^2 + \cdots = a_0 x + a_1 x^2 + a_2 x^3 + \cdots$$

Then

$$a_2 = 0$$
$$3\cdot 2\, a_3 = a_0$$
$$4\cdot 3\, a_4 = a_1$$
$$5\cdot 4\, a_5 = a_2$$
$$6\cdot 5\, a_6 = a_3$$
$$7\cdot 6\, a_7 = a_4$$
$$\cdots$$

$$0 = a_2 = a_5 = a_8 = \cdots$$

$$a_3 = \frac{a_0}{3\cdot 2}, \quad a_6 = \frac{a_0}{(6\cdot 5)(3\cdot 2)}, \ldots$$

$$a_4 = \frac{a_1}{4\cdot 3}, \quad a_7 = \frac{a_1}{(7\cdot 6)(4\cdot 3)}, \ldots$$

Moreover $y(0) = 7 = a_0$ and $y'(0) = 9 = a_1$

Thus

$$y(x) = 7\left(1 + \frac{x^3}{3\cdot 2} + \frac{x^6}{(6\cdot 5)(3\cdot 2)} + \cdots\right) + 9\left(x + \frac{x^4}{4\cdot 3} + \frac{x^7}{(7\cdot 6)(4\cdot 3)} + \cdots\right)$$

(Q6b, 3pts) Find the radius of convergence of the P.S. solution you found in part (a).

$p = 0$ or $q(x) = -x$ both analytic with $R = \infty$

Therefore by Theorem $R = \infty$

(Q7a, 5pts) Evaluate the following using the Fourier transform table provided. Expain each step and cite the entries you use.

$$F\left\{\frac{\sin 2x}{4x^2+3}\right\}$$

We want to use rule 14 by setting $f(x) = \frac{1}{4x^2+3}$

$$F\{f(x)\sin(2x)\} = \frac{1}{2i}\left[\hat{f}(w-2) - \hat{f}(w+2)\right]$$

need $\hat{f}$: $= F\left\{\frac{1}{4x^2+3}\right\} = \frac{1}{4} F\left\{\frac{1}{x^2+3/4}\right\} = \frac{\pi}{2\sqrt{3}} e^{-\frac{\sqrt{3}}{2}|w|}$

by rule 1

Thus answer is $\boxed{\frac{\pi}{i4\sqrt{3}}\left[e^{-\frac{\sqrt{3}}{2}|w-2|} - e^{-\frac{\sqrt{3}}{2}|w+2|}\right]}$

(Q7b, 5pts) Evaluate the following using the Fourier transform table provided. Expain each step and cite the entries you use.

$$F^{-1}\left\{we^{-3w^2}\right\}$$

Want to use rule 19:

$$F^{-1}\{we^{-3w^2}\} = \frac{1}{i}F^{-1}\{(iw)e^{-3w^2}\} = \frac{1}{i}\frac{d}{dx}\left[F^{-1}\{e^{-3w^2}\}\right]$$

$$= \frac{1}{i}\frac{d}{dx}\left[\frac{1}{2\sqrt{3\pi}}e^{-x^2/12}\right]$$ by rule 6

$$= \frac{-x}{i12\sqrt{3\pi}}e^{-x^2/12}$$

(Q8, 10pts) Solve the following problem using separation of variables,

$$2u_{xx} - u_t = 0, \qquad x \in (0, \pi),\ 0 < t$$

$$\begin{cases} u_x(0,t) = 0, & u_x(\pi,t) = 0, \quad 0 < t \\ u(x,0) = x, & \qquad\qquad 0 < x < \pi \end{cases}$$

Note that $L = \pi$. It is OK to leave the coefficients in integral form.

Using sep. of variables as before

$$u = H + Ix + \left[J(\cos kx) + K\sin(kx)\right] e^{-2k^2t}$$

$$u_x(0,t) = 0 = I + Kke^{-2k^2t} \Rightarrow I = 0,\ K = 0$$

$$u_x(\pi,t) = 0 + 0 - Jk\sin(k\pi)e^{-2k^2t} \qquad \text{Since } I = K = 0.$$

$$\Rightarrow k\pi = n\pi \qquad n = 1,2,3$$

$$n = k = 1,2,3,\dots$$

Thus

$$\boxed{u(x,t) = H + \sum_{n=1}^{\infty} J_n \cos(nx)\, e^{-2n^2t}}$$

IC:

$$u(x,0) = x = \underbrace{H + \sum_{n=1}^{\infty} J_n \cos(nx)}_{\text{this is HRC}} \qquad 0 < x < \pi$$

$$\boxed{H = \frac{1}{\pi}\int_0^{\pi} x\,dx = \pi/2}$$

$$\boxed{J_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx = \dots}$$

(Q9) Consider heat distribution problem on a rod. Recall that if a steady-state solution $u_s(x)$ exists then it can be determined by

$$u_s(x) = \lim_{t\to\infty} u(x,t).$$

In each case below, for which values of c the steady-state solution exists ? If it exists then calculate u_s.
Hint: You don't need to solve the heat equation. Also you might find conservation principle useful. In order to obtain it you can start with $\int_0^1 u_{xx}dx = \int_0^1 u_t dx = \frac{d}{dt}\int_0^1 u dx$.
(Q9a, 5pts)

$$u_{xx} - u_t = 0, \qquad x \in (0,1),\ 0 < t$$

$$\begin{cases} u(0,t) = 1, & u_x(1,t) = c, \quad 0 < t \\ u(x,0) = 11, & 0 < x < 1 \end{cases}$$

steady-solution exists if $u_s = ax+b$ solves $u_s'' = 0$, $u_s(0) = 1$, $u_s'(1) = c$

$u_s(0) = 1 = b \Rightarrow b = 1$

$u_s'(1) = c = a \Rightarrow a = c$

Thus $\boxed{u_s(x) = cx+1}$ exists for all c.

(Q9b, 5pts)

$$u_{xx} - u_t = 0, \qquad x \in (0,1),\ 0 < t$$

$$\begin{cases} u_x(0,t) = 1, & u_x(1,t) = c, \quad 0 < t \\ u(x,0) = 11, & 0 < x < 1 \end{cases}$$

Similarly $u_s = ax+b$ $\quad u_s'(0) = 0 = a \Rightarrow a=1$, $\quad u_s'(1) = c = a \Rightarrow a = c$ $\quad$ we need $\boxed{c=1=a}$

Then $u_s = x+b$ and $c=1$

using hint: $\int_0^1 u_{xx}\,dx = u_x(1,t) - u_x(0,t) = 1-1 = 0$ for all t.

Thus $0 = \frac{d}{dt}\int_0^1 u\,dx \Rightarrow \int_0^1 u\,dx$ is constant.

$11 = \int_0^1 u(x,0)\,dx = \int_0^1 u_s(x)\,dx = \int_0^1 (x+b)\,dx = \frac{1}{2} + b \Rightarrow \boxed{b = \frac{21}{2}}$

$\boxed{u_s(x) = x + 21/2}$

(Q10a, 7pts) Find all separable solutions (physical and non-physical) of

$$y_{xx} = y_{tt} + y.$$

Recall y is separable if it can be written as $y(x,t) = X(x)T(t)$.

$$\frac{X''}{X} = \frac{T''}{T} + 1 = \beta \leftarrow \text{constant}$$

$\underline{X'' = \beta X} \Rightarrow$ when
- $\beta = 0$: $X = X_1(x) = Ax + B$
- $\beta < 0$: $X = X_2(x) = C\cos\sqrt{|\beta|}x + D\sin\sqrt{|\beta|}x$
- $\beta > 0$: $X = X_3(x) = Ee^{\sqrt{|\beta|}x} + Fe^{-\sqrt{|\beta|}x}$

$\underline{T'' = (\beta-1)T}$: when
- $\beta = 1$: $T = T_1(x) = G + Ht$
- $\beta < 1$: $T = T_2(t) = I\cos\sqrt{|\beta-1|}t + J\sin(\sqrt{|\beta-1|}t)$
- $\beta > 1$: $T = T_3(t) = Ke^{\sqrt{|\beta-1|}t} + Le^{-\sqrt{|\beta-1|}t}$

Thus 5 cases:

$\beta < 0$: $X = X_2 \cdot T_2$	$0 < \beta < 1$: $X = X_3 T_2$
$\beta = 0$: $X = X_1 \cdot T_2$	$\beta = 1$: $X = X_3 T_1$
	$\beta > 1$: $X = X_3 T_3$

alternative soln

(Q10a, 7pts) Find all separable solutions (physical and non-physical) of

$$y_{xx} = y_{tt} + y.$$

Recall y is separable if it can be written as $y(x,t) = X(x)T(t)$.

$$\frac{X''}{X} = \frac{T''}{T} + 1 = \beta \leftarrow \text{constant}$$

$$X'' = \beta X \Rightarrow X(x) = \begin{cases} Ax + B & \beta = 0 \\ Ce^{\sqrt{\beta}x} + De^{-\sqrt{\beta}x} & \beta \neq 0 \end{cases}$$

$$T'' = (\beta-1)T \Rightarrow T(t) = \begin{cases} Ex + D & \beta = 1 \\ Fe^{\sqrt{\beta-1}t} + Ge^{-\sqrt{\beta-1}t} & \beta \neq 1 \end{cases}$$

Note
we allow $\sqrt{\beta}$ to be complex with $\sqrt{-1} = i$

$\sqrt{\beta-1}$ is complex

Thus 3 cases

$\beta = 0$: $y(x,t) = (Ax+B)(Fe^{\sqrt{\beta-1}t} + Ge^{-\sqrt{\beta-1}t}) \rightarrow \sqrt{\beta-1} = i$

$\beta = 1$: $y(x,t) = (Ce^{\sqrt{\beta}x} + De^{-\sqrt{\beta}x})(Ex + D) \rightarrow \sqrt{\beta} = 1$

otherwise : $y(x,t) = (Ce^{\sqrt{\beta}x} + De^{-\sqrt{\beta}x})(Fe^{\sqrt{\beta-1}t} + Ge^{-\sqrt{\beta-1}t})$

(Q10b, 7pts) Solve the following vibrating string problem using separation of variables.

$$c^2 y_{xx} = y_{tt}, \qquad x \in (0, L),\ 0 < t$$

$$\begin{cases} y(0,t) = 0, & y_x(L,t) = 0, \quad 0 < t \\ y(x,0) = f(x), & y_t(x,0) = 0, \quad 0 < x < L \end{cases}$$

Letting $y = XT$ and solving X and T as before :

$y(x,t) = A + Bx)(H + It) + (D\cos kx + E\sin kx)(J\cos kct + K\sin kct)$

BC: $y(0,t) = 0 = A(H + It) + (D + 0)(J\cos kct + K\sin kct)$

$\Rightarrow A = 0,\ D = 0$ we keep J, K free.

then $y(x,t) = Px + Qtx + B\sin kx\,(R\cos kct + S\sin(kct))$

here we replaced constants for brevity.

$$y_x(L,t) = 0 = P + Qt + \kappa \cos \kappa L \left[R \cos \kappa c t + S \sin \kappa c t \right]$$

$$\Rightarrow \quad P = 0 \, , \quad Q = 0$$

$$\kappa L = \frac{\pi}{2} , \frac{3\pi}{2} , \ldots = \frac{n\pi}{2} \qquad n = 1, 3, \ldots$$

$$\kappa = \frac{n\pi}{2L} \quad \text{where } n = 1, 3, \ldots$$

<u>Thus</u>

$$y(x,t) = \sum_{n=1,3,\ldots} \sin \frac{n\pi x}{2L} \left[R_n \cos \frac{n\pi c t}{2L} + S_n \sin \frac{n\pi c t}{2L} \right]$$

<u>IC</u>: $\quad y(x,0) = f(x) = \sum_{n=1,3,\ldots} R_n \sin \frac{n\pi x}{2L}$

from QRS: $\quad \boxed{R_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{2L} \, dx}$

$$y_t(x,0) = 0 = \sum_{n=1,3,\ldots} S_n \frac{n\pi c}{2L} \cos \frac{n\pi c t}{2L} \Rightarrow \boxed{S_n = 0}$$

<u>Thus</u>

$$\boxed{y(x,t) = \sum_{n=1,3,\ldots} R_n \sin \frac{n\pi x}{2L} \cos \frac{n\pi c t}{2L}}$$

with R_n defined as above