

# Strategic Effects of Renegotiation-Proof Contracts\*

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## Abstract

It is well known that non-renegotiable contracts with third parties may have an effect on the outcome of a strategic interaction and thus serve as a commitment device. We address this issue when contracts are renegotiable. More precisely, we analyze the equilibrium outcomes of two-stage games with renegotiation-proof third-party contracts in relation to the equilibrium outcomes of the same game without contracts. We assume that one of the parties in the contractual relationship is unable to observe everything that happens in the game when played by the other party. We first show that when contracts are non-renegotiable, the set of equilibrium outcomes of the game with contracts is restricted to a subset of Nash equilibrium outcomes of the original game. Introducing renegotiation, in general, imposes further constraints and in some games implies that only subgame perfect equilibrium outcomes of the original game can be supported. However, there is a large class of games in which non-subgame perfect equilibrium outcomes can also be supported, and hence, third-party contracts still have strategic implications even when they are renegotiable.

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# 1 Introduction

As it was so eloquently illustrated by Schelling (1960), contracts with third parties may have an effect on the outcome of a strategic interaction and therefore could be used as a commitment device. Under the assumption that contracts are *observable and non-renegotiable*, the previous literature has formally illustrated this possibility in many settings. Vickers (1985), Fershtman and Judd (1987), and Sklivas (1987) analyze the effects of managerial compensation contracts on product market competition, and show that such contracts can provide a strategic advantage.<sup>1</sup> Brander and Lewis (1986) do the same for debt contracts, whereas Bolton and Scharfstein (1990) and Snyder (1996) study optimal financial contracts when there is a threat of predation by a “deep-pocket” incumbent. Spencer and Brander (1983), Brander and Spencer (1985), and Eaton and Grossman (1986) study strategic design of trade and industrial policies when firms compete in international markets.

Each one of these models falls into one of two possible categories of games that third-party contracts may induce. In *delegation games*, a player signs a contract that specifies an outcome contingent transfer to an agent, who in turn plays the game in place of the (principal) player. For example, in Fershtman and Judd (1987) the owner of a firm signs a compensation contract with a manager, who in turn chooses the output level in the Cournot game that follows. In *games with side contracts*, the player signs a contract with a third-party but does not delegate the play of the game. In Brander and Lewis (1986), for example, the firm signs a debt contract with a lender and then participates in quantity competition.

Fershtman, Judd, and Kalai (1991), Polo and Tedeschi (2000), and Katz (2006) prove different “folk theorems” for some classes of delegation games under observable and non-renegotiable contracts.<sup>2</sup> The effects of *unobservable and non-renegotiable* third-party contracts are also well-understood. Within the context of delegation games, Katz (1991) showed that the Nash equilibrium outcomes of a game with and without delegation are identical. Koçkesen and Ok (2004) and Koçkesen (2007) addressed the same question within the context of extensive form games and showed that all (and only) Nash equilibrium outcomes of the original game can be supported as a sequential equilibrium outcome of the delegation game. In particular, they showed that outcomes that are not subgame perfect in the original game may arise as a sequential equilibrium outcome of the induced delegation game, i.e., unobservable contracts may have a strategic effect as long as they are non-renegotiable.<sup>3</sup>

Non-renegotiable contracts yield equilibrium outcomes that are not subgame perfect in the original game by inducing suboptimal behavior (from the perspective of the preferences in the original game) at certain points in the game. These points must be off the equilibrium path, since otherwise the player and the third party could increase the total surplus available to them by inducing optimal play. Therefore, if the game ever reaches such a point, they will have an incentive to renegotiate the existing contract. This implies that, if renegotiation takes place without any friction, only the subgame

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<sup>1</sup>Koçkesen, Ok, and Sethi (2000) extend these results to more general classes of games.

<sup>2</sup>Prat and Rustichini (2003) and Jackson and Wilkie (2005) analyze related models in which players can write action contingent contracts before the game is played. However, in Prat and Rustichini (2003) there are multiple principals and agents and principals can contract with any agent, whereas in Jackson and Wilkie (2005) any player can write a contract with any other. Unlike in the literature mentioned in the text, in these papers contractual relationships are not exclusive and the focus is on the efficiency properties of the equilibrium set. Also related is Bhaskar (2009), in which players need to pay a price to a supplier in order to play certain actions that are controlled by this supplier.

<sup>3</sup>Using an example, Katz (1991) also showed that the equivalence between the equilibrium outcomes of games with and without delegation does not hold if one uses refinements of Nash equilibrium. Likewise, Fershtman and Kalai (1997) showed that any outcome of an ultimatum bargaining game can be supported as a trembling hand perfect equilibrium.

perfect equilibrium outcomes of the original game can be supported. In other words, renegotiable third-party contracts have no strategic effect.

Therefore, the question at hand becomes interesting only when there are frictions in the renegotiation process. In this paper we analyze the strategic design of *unobservable and renegotiable* third-party contracts in an environment where such a friction arises quite naturally: We assume that the player who does not participate in the actual play of the game – the principal in delegation games and the third party in games with side contracts – is unable to observe every outcome of the game.<sup>4</sup> If we further assume that such unobserved outcomes remain unverifiable throughout, then contracts can be made contingent only on a partition of the set of outcomes of the underlying game. For example, a bank may be able to observe and contract upon only the level of capacity expansion made by the firm to which it lends, but not those made by this firm's competitors. Similarly, a seller may be able to observe and contract on whether an item has been sold by his agent or not, but not the exact price at which the transaction has occurred; or a government can observe and subsidize/tax only the production level of its domestic firm, but not that of the foreign competitor, etc. In all these scenarios, the player who actually plays the game may not be able to credibly signal the existence of a mutually beneficial contract and his renegotiation attempt may fail. Motivated by this observation we ask and answer the following question in the current paper: *Which outcomes can be supported in games with unobservable and renegotiable third-party contracts?*

For reasons of analytical tractability and expositional clarity, we limit our analysis to finite two-stage games: Player 1 moves first by choosing an action  $a_1 \in A_1$ , and after observing  $a_1$ , player 2 chooses an action  $a_2 \in A_2$ . Let us call this game the *original game*. In the induced *game with third-party contracts*, player 2 and a third party agree on a contract  $f : A_2 \rightarrow \mathbb{R}$ , which specifies a transfer between them as a function of  $a_2$ . If this is a delegation game, then the contract also transfers the right to choose the action  $a_2$  to the agent, whereas if it is a game with side contracts, player 2 retains the right to choose  $a_2$ . Let us call the player who has the right to choose  $a_2$  the “active player”, and the other one the “passive player”. We assume that  $a_1$  cannot be observed by the passive player at any time and is unverifiable. Therefore, feasible contracts specify the transfer as a function of  $a_2$ , rather than  $(a_1, a_2)$ . We also assume that the contract that the parties agree upon is unobservable to player 1, which is only natural since we do not want to impose any physical restriction on renegotiation opportunities.

Once the contracting stage is completed, player 1 chooses an action  $a_1$ . The active player observes  $a_1$  and decides whether to end the game by choosing an action  $a_2$  or offer a new contract  $g : A_2 \rightarrow \mathbb{R}$  to the passive player. The passive player has to decide whether to accept  $g$  or not, without being informed about  $a_1$ . If he accepts  $g$ , then the active player chooses  $a_2$  and the payoffs are determined according to  $g$ , otherwise they are determined according to  $f$ . Our objective is to characterize the set of outcomes of the original game that can be supported in a perfect Bayesian equilibrium (PBE) of the induced game with (renegotiation-proof) third-party contracts.

It is crucial that  $a_1$  is unverifiable, in addition to being unobservable, during the renegotiation stage. Otherwise, renegotiation would again imply that only the subgame perfect equilibrium outcomes of the original game can be supported. To see this, suppose that  $a_1$ , although unobservable to the passive player before  $a_2$  is chosen, becomes verifiable after the game ends and hence can be

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<sup>4</sup>Katz (1991) was the first to consider this scenario within the context of an ultimatum bargaining game and provided the initial motivation for this research.

contracted upon. Also suppose that after an action  $\hat{a}_1 \in A_1$ , the optimal action is  $\hat{a}_2 \in A_2$ , but the original contract induces the active player to choose a suboptimal action. In this case, the active player can offer a new contract  $g(a_1, a_2)$  after  $\hat{a}_1$  that punishes himself severely if the outcome is not  $(\hat{a}_1, \hat{a}_2)$ , and shares the extra surplus that results from playing optimally if the outcome is  $(\hat{a}_1, \hat{a}_2)$ . Since such a contract can reasonably be offered only after  $\hat{a}_1$ , the passive player should accept it. In other words, renegotiation-proof contracts cannot support suboptimal actions after any  $a_1$ .

For the lack of a better term, we denote the fact that transfers cannot be conditioned upon the entire outcome space as contracts being incomplete.<sup>5</sup> As we argued in the previous paragraph, in our setting, this incompleteness is a necessary condition for supporting outcomes that are not subgame perfect in the original game. However, contract incompleteness itself brings about interesting issues that are independent of the existence of renegotiation opportunities. Supporting an outcome in an equilibrium of the game with third-party contracts depends on the ability of writing a contract that gives proper incentives to the active player to play certain strategies. When contracts are complete, as in Koçkesen and Ok (2004) and Koçkesen (2007), finding such contracts is relatively easy, as incentive compatibility does not arise as a binding constraint. When contracts are incomplete, however, only incentive compatible strategies can be supported. We analyze this question in section 5.1 and show that, if payoff functions exhibit increasing differences, then only (and all) the Nash equilibria of the original game in which player 2's strategy is increasing can be supported.

As we show in section 5.2, renegotiation imposes further constraints on outcomes that can be supported. In that section, we completely characterize contract-strategy pairs that are renegotiation-proof and give necessary and sufficient conditions for a strategy to be renegotiation-proof. In section 6 we apply our results to an environment that is common to many economically relevant games, such as the Stackelberg, sequential Bertrand, relationship specific investment (hold-up), and ultimatum bargaining games, and completely characterize the set of outcomes that can be supported with incomplete and renegotiation-proof contracts. In the same section we also identify an environment in which renegotiation has no bite, i.e., the set of outcomes that can be supported with incomplete and renegotiable contracts is the same as the one that can be supported with incomplete and non-renegotiable contracts.

Previous literature has identified two scenarios in which renegotiable contracts may have a commitment value: (1) games in which there is exogenous asymmetric information between the player and the third party (Dewatripont (1988) and Caillaud, Jullien, and Picard (1995)); and (2) two-stage games with nontransferable utilities (Bensaid and Gary-Bobo (1993)).

Dewatripont (1988) analyzes an entry-deterrence game in which the incumbent signs a contract with a labor union before the game begins. A potential entrant observes the contract and then decides whether to enter or not. Renegotiation takes place after the entry decision is made, during which the union offers a new contract to the incumbent, who by this time has received some payoff relevant private information. The paper shows that commitment effects exist in such a model and may deter entry. As noted by Bolton and Dewatripont (2005, pp. 630-636), there are two limitations to this analysis: first, it is assumed that renegotiation can take place only after entry and after the incumbent has received the private information; second, this model does not address the effects of third party contracts in other interesting settings, for example in oligopoly models in which competition between

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<sup>5</sup>This may go against the contract theory jargon in which a contract is regarded incomplete only if the transfer cannot be conditioned upon an *observable* but unverifiable contingency.

firms is of Bertrand type. We overcome these limitations by analyzing arbitrary two-stage games, and allowing secret renegotiation at any stage of the game. Another difference between our model and Dewatripont's is that in his model the friction in the renegotiation process arises from exogenously given asymmetric information, whereas in ours it comes from the inability of the passive player to observe player 1's move. These differences turn out to be significant, as we show in Section 6.1 that, in our model, renegotiable contracts cannot deter entry.

Caillaud et al. (1995) analyze a game between two principal-agent hierarchies. Each principal-agent pair faces a classical hidden knowledge problem. In the first stage of the game (public contracting stage) each principal decides whether to publicly offer a contract to the agent; in the second stage (secret renegotiation stage) each principal offers a secret contract to the agent that, if accepted, overwrites the public contract that might have been offered in stage 1; in the third stage (the market game stage) each agent receives payoff relevant information, decides whether to quit, and if he does not quit, he plays a normal form game with the other agent. Their main question is whether there exist equilibria of this game in which the principals choose not to offer a public contract in stage 1. If the answer to this question is no, then the interpretation is that contracts have commitment value. They show that contracts have commitment value if the market game stage is of Cournot type, but not if it is of Bertrand type.

To understand how our paper complements the literature, it is important to notice that our approach is to compare the equilibrium outcomes of a principals-only game, with the equilibrium outcomes of the same game with third-party contracts. In Caillaud et al. (1995), however, the benchmark model is not a principals-only game, but a competing hierarchies model, where each principal must design a contract facing an asymmetric information problem with the agent and the strategic interaction problem with the other principal-agent pair. Given their approach, renegotiable contracts have commitment value if a principal finds it profitable to commit to an increased reservation utility of the agent, which is achieved by offering a public contract in the first stage. In contrast, our results do not depend on whether contracts are public or not, as long as they can be renegotiated later on.<sup>6</sup> In Section 6.1, we further compare and contrast our model and results with theirs within the context of Cournot and Bertrand games.

Finally, Bensaïd and Gary-Bobo (1993) analyze a model in which the original game is a two-stage game and the initial contract can be renegotiated after player 1 chooses an action. However, in their model player 1's action is contractible and observable, but utility is not transferable between player 2 and the third-party. They show that, in a certain class of games, contracts with third parties have a commitment effect, even when they are renegotiable.<sup>7</sup>

The next two sections lay out the model and the question addressed in this paper. Section 4 presents two simple games, one of which illustrates that non-subgame perfect outcomes can be supported with incomplete and renegotiable contracts, while the other one shows that this is not true in general. Therefore, characterization of equilibrium outcomes that can be supported with such contracts seems to be an interesting matter. Sections 5 and 6 deal with this question in general two-stage games and Section 7 does the same using intuitive criterion (Cho and Kreps (1987)) as the equilibrium concept. Section 8 concludes with some remarks and open questions, while section 9 contains

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<sup>6</sup>Katz (1991), on page 308, argues that in many cases it is impossible to observe the actual contract that governs the incentives of the parties.

<sup>7</sup>Similarly, Snyder (1996) studies the commitment effects of renegotiable financial contracts in a model with non-transferable utility, where the non-transferability arises from capital market imperfections.

the proofs of our results.

## 2 The Model

Our analysis starts with a two-player extensive form game, which we call the *original game*. We then allow one of the players to sign a contract with a third party before the game begins and call this new game the *game with third-party contracts*. The contracts specify a transfer between the player and the third party as a function of the contractible outcomes of the original game. After the contract is signed, the game itself may be played by either the third party, in which case we have a delegation game, or by the original player herself, in which case we have a game with side contracts. Although, for the sake of concreteness, we will use the framework of games with side contracts, our main results go through for delegation games as well.

We aim to characterize the equilibrium outcomes of the game with third party contracts in relation with those of the original game. We are particularly interested in whether the induced game with contracts has equilibrium outcomes that are not equilibrium outcomes in the original game, i.e., whether third-party contracts “matter”.

The main intuition behind our results is best seen in a simple model in which the original game has only two stages: Player 1 moves first and player 2 second. Limiting player 1’s move to only the first stage makes formulating the model, e.g., introducing an order structure on the set of histories in the game and defining increasing differences, much easier and renders the results more transparent.

Limiting the analysis to two-stage games simplifies the analysis further as we may, without loss of generality, assume that only the second mover can sign contracts with third parties. Third-party contracts introduce equilibrium outcomes that are not equilibrium outcomes in the original game by inducing sequentially irrational play (from the perspective of the preferences in the original game) at information sets that are not reached in equilibrium. Since player 1 moves only once, at the beginning of the game, allowing him to contract as well would not change the set of equilibrium outcomes at all.<sup>8</sup>

In light of these observations, we define the *original game*, denoted  $G$ , as a two-stage game: Player 1 chooses  $a_1 \in A_1$ , and player 2, after observing  $a_1$ , chooses  $a_2 \in A_2$ , where  $A_1$  and  $A_2$  are finite sets. Payoff function of player  $i \in \{1, 2\}$  is given by  $u_i : A \rightarrow \mathbb{R}$ , where  $A = A_1 \times A_2$ .

The *game with incomplete and non-renegotiable third-party contracts*, denoted  $\Gamma(G)$ , is a three player extensive form game described by the following sequence of events:

**Stage I.** Player 2 offers a contract  $f : A_2 \rightarrow \mathbb{R}$  to a third party.

**Stage II.** The third party accepts (denoted  $y$ ) or rejects (denoted  $n$ ) the contract.

1. In case of rejection the game ends, the third party receives a fixed payoff of  $\delta \in \mathbb{R}$ , and player 1 and 2 receive  $-\infty$ .<sup>9</sup>
2. In case of acceptance, the game goes to Stage III.

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<sup>8</sup>Of course, as it was shown in Koçkesen (2007), in games with more than two stages this is not the case.

<sup>9</sup>Alternatively, we could assume that if the third party rejects an offer, then the original game is played without a contract. However, this assumption introduces additional notation and technical details without changing our results in any substantive way.

**Stage III.** Player 1 chooses an action  $a_1 \in A_1$  (without observing the contract), player 2 observes  $a_1$ .

**Stage IV.** Player 2 chooses an action  $a_2 \in A_2$ .

Since we assume that if the contract offer is rejected, the game ends and players 1 and 2 receive very small payoffs, the contract offer is accepted in all equilibria. Therefore, we will, for the sake of notational simplicity, denote the set of outcomes as  $Z = \mathcal{C} \times A$ , where  $\mathcal{C} = \mathbb{R}^{A_2}$  is the contract space. For any outcome  $(f, a) \in Z$  the payoff functions are given by

$$\begin{aligned} v_1(f, a) &= u_1(a_1, a_2) \\ v_2(f, a) &= u_2(a_1, a_2) - f(a_2) \\ v_3(f, a) &= f(a_2) \end{aligned}$$

where  $v_3$  is the payoff function of the third party.

The above formulation assumes that after the contract is signed, it is player 2 who actually plays the game, i.e., we have a game with side contracts. In a delegation game, it is the third party who plays the game, in which case player 2's payoff function would be given by  $f(a_2)$ , and the third party's by  $u_2(a_1, a_2) - f(a_2)$ .<sup>10</sup>

We also assume that player 2 has the entire bargaining power during the contracting phase. This assumption has no effect on our results regarding the set of equilibrium outcomes, but clearly has implications regarding the equilibrium payoff of player 2. Also note that  $\delta$  could represent either the outside option of the third party, such as that of an agent or a lender, or could be used to model some other constraint on the transfers, such as the upper bound on the amount of export subsidy.

The game is with renegotiable contracts if the contracting parties can renegotiate the contract after Stage III and before Stage IV. We assume that renegotiation can be initiated only by the player who actually plays the game. However, as it will become apparent after we introduce our concept of renegotiation-proofness, the results remain intact if the renegotiation process is initiated by the other player. The following sequence of events describe the renegotiation process after any history  $(f, a_1)$ .

**Stage III(i).** Player 2 either offers a new contract  $g \in \mathcal{C}$  to the third party or chooses an action  $a_2$ . In the latter case the game ends and the outcome is  $(f, a)$ .

**Stage III(ii).** If player 2 offers a new contract, the third party (without observing  $a_1$ ) either accepts (denoted  $y$ ) or rejects (denoted  $n$ ) the offer.

If the third party rejects the renegotiation offer  $g$ , then player 2 chooses  $a_2 \in A_2$  and the outcome is payoff equivalent to  $(f, a)$ . If he accepts, then player 2 chooses  $a_2 \in A_2$  and the outcome is payoff equivalent to  $(g, a)$ . This completes the description of the *game with incomplete and renegotiable contracts*, which we denote as  $\Gamma_R(G)$ .

A *behavior strategy* for player  $i \in \{1, 2, 3\}$  is defined as a set of probability measures  $\beta_i \equiv \{\beta_i[I] : I \in \mathcal{I}_i\}$ , where  $\mathcal{I}_i$  is the set of information sets of player  $i$  and  $\beta_i[I]$  is defined on the set of actions

<sup>10</sup>In a delegation game, this payoff specification would be reasonable if the third party can inherit player 2's preferences once the game is delegated to him. Consider, for example, a seller who delegates the sale of an item to an agent and suppose that she cannot observe the actual price at which the item is sold. In this case the contract would specify a payment from the agent to the seller contingent upon whether a sale has occurred or not. If the seller and the agent care only about money and are risk neutral, then the above payoff specification would indeed be the appropriate one.

available at information set  $I$ . One may write  $\beta_i[h]$  for  $\beta_i[I]$  for any history  $h \in I$ . By a *system of beliefs*, we mean a set  $\mu \equiv \{\mu[I] : I \in \mathcal{I}_i \text{ for some } i\}$ , where  $\mu[I]$  is a probability measure on  $I$ . A pair  $(\beta, \mu)$  is called an *assessment*. An assessment  $(\beta, \mu)$  is said to be a *perfect Bayesian equilibrium* (PBE) if (1) each player's strategy is optimal at every information set given her beliefs and the other players' strategies; and (2) beliefs at every information set are consistent with observed histories and strategies.<sup>11</sup>

### 3 The Query

We will limit our analysis to pure behavior strategies, and hence a strategy profile of the original game  $G$  is given by  $(b_1, b_2) \in A_1 \times A_2^{A_1}$ . For any behavior strategy profile  $(b_1, b_2)$  in  $G$ , we say that an assessment  $(\beta, \mu)$  in  $\Gamma(G)$  *induces*  $(b_1, b_2)$  if in  $\Gamma(G)$  player 1 plays according to  $b_1$  and, after the equilibrium contract, player 2 plays according to  $b_2$ . Note that in  $\Gamma_R(G)$ , player 2 may choose an action  $a_2 \in A_2$  either without renegotiating the initial contract or after attempting renegotiation.

We restrict our attention to equilibria in which the equilibrium contract is not renegotiated. As Beaudry and Poitevin (1995) point out, this is necessary for renegotiation to have any bite, as one can always replicate an equilibrium outcome of the game without renegotiation by making player 2 offer an initial contract that is accepted only because it is going to be renegotiated later on.<sup>12</sup> This leads to the following definition.

**Definition 1** (Renegotiation-Proof Equilibria). A perfect Bayesian equilibrium  $(\beta^*, \mu^*)$  of  $\Gamma_R(G)$  is *renegotiation-proof* if the equilibrium contract is not renegotiated after any  $a_1$ .

Note that the set of renegotiation-proof equilibria is actually a subset of perfect Bayesian equilibria in which the equilibrium contract is not renegotiated. The latter would be defined so that the equilibrium contract is not renegotiated after any action of player 1 that gives him a higher payoff under a renegotiated contract than the equilibrium payoff. However, working with this weaker notion of renegotiation-proofness would only introduce additional complexity into our presentation without changing the main results in any interesting way.

**Definition 2.** A strategy profile  $(b_1, b_2)$  of the original game  $G$  can be *supported with incomplete and non-renegotiable contracts* if there exists a perfect Bayesian equilibrium of  $\Gamma(G)$  that induces  $(b_1, b_2)$ . Similarly, a strategy profile  $(b_1, b_2)$  of the original game  $G$  can be *supported with incomplete and renegotiable contracts* if there exists a renegotiation-proof perfect Bayesian equilibrium of  $\Gamma_R(G)$  that induces  $(b_1, b_2)$ .

Our main query can therefore be phrased as follows:

*Which outcomes of a given original game can be supported with incomplete and renegotiable (or non-renegotiable) contracts?*

### 4 Examples

In this section we analyze two simple games, an ultimatum bargaining and a sequential battle-of-the-sexes game, each of which has a unique subgame perfect equilibrium. We will show that renegotiable

<sup>11</sup>See Fudenberg and Tirole (1991) for a precise definition of perfect Bayesian equilibrium.

<sup>12</sup>See also Maskin and Tirole (1992) on this point.

contracts can support a Nash equilibrium outcome that is not perfect in the bargaining game, while only the subgame perfect equilibrium outcome can be supported in the battle-of-the-sexes game.

#### ULTIMATUM BARGAINING

Consider a simple ultimatum bargaining game in which player 1 moves first, by choosing the action  $L$  or  $R$ , after which player 2 moves by choosing  $l$  or  $r$ . The payoffs corresponding to each outcome are given in the game tree in Figure 1, where the first number is player 1's payoff and the second number player 2's.

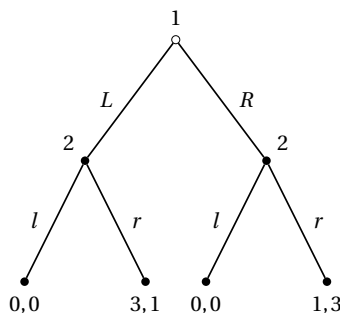


Figure 1: Ultimatum Bargaining Game

The unique subgame perfect equilibrium (SPE) of this game is  $(L, rr)$ , i.e., player 1 plays  $L$  and player 2 plays  $r$  after both  $L$  and  $R$ . There is another Nash equilibrium of this game given by  $(R, lr)$ . This equilibrium gives player 2 a higher payoff than does the subgame perfect equilibrium, and hence if she could commit to the strategy  $lr$  in a credible way she would do so.

The set of perfect Bayesian equilibrium outcomes of the induced game with third-party contracts differs depending upon the characteristics of the contracts. If contracts are observable, non-renegotiable, and complete, in the sense that the transfers can be made conditional on the entire set of outcomes, then the unique PBE outcome of the game is  $(R, r)$ . A contract that pays the third party  $\delta$  if the outcome is  $(L, l)$  or  $(R, r)$  and pays more than  $1 + \delta$  otherwise is a possible equilibrium contract that achieves this outcome. This is nothing but another illustration of the commitment value of observable, non-renegotiable, and complete contracts.

If contracts are unobservable, then the SPE outcome of the original game, i.e.,  $(L, r)$ , is also an equilibrium outcome of the game, in addition to  $(R, r)$ . This is an example illustrating the main results in Koçkesen and Ok (2004) and Koçkesen (2007), which state that all Nash equilibrium outcomes can be supported with unobservable (but complete and non-renegotiable) contracts.

If contracts can be renegotiated after the game begins, but they are complete, then the unique equilibrium outcome of the delegation game is the SPE outcome of the original game, irrespective of whether contracts are observable or unobservable. The reason is simple: The only way a non-SPE outcome can be supported is through player 2 playing  $l$  after player 1 plays  $L$ , which is sequentially irrational from the perspective of player 2's preferences in the original game. Therefore, if player 1 plays  $L$ , player 2 and the third party have an incentive to renegotiate the contract so that under the new contract player 2 plays  $r$ . In other words, in any PBE, player 2 must play  $r$  after any action choice of player 1, and hence player 1 must play  $L$ .

The conclusion is entirely different if only player 2's action is contractible and observable by the

third party. In this case, the non-SPE outcome  $(R, r)$  is an equilibrium outcome of the induced game with third-party contracts, even if these contracts can be renegotiated.<sup>13</sup>

The following is a PBE of this game that supports this outcome. Player 2 offers the contract  $f$  that transfers  $\delta$  to the third party if she plays  $r$ , and transfers  $\delta - 1$  if she plays  $l$ . The third party accepts any contract that gives him an expected payoff of at least  $\delta$ ; player 1's beliefs put probability 1 on  $f$  and he plays  $R$ ; player 2 chooses not to renegotiate  $f$  and plays  $l$  following  $L$  and  $r$  following  $R$ . In the event of an out-of-equilibrium renegotiation offer after  $f$ , the third party believes that player 1 has played  $R$  and rejects any contract that transfers him less than  $\delta$ . Note that in this equilibrium player 2's payoff is  $3 - \delta$ , which implies that as long as  $\delta < 2$ , player 2 prefers to sign such a contract even if she has the option of playing the game without a contract.

Few remarks are in order about this example. First, notice that, in the above equilibrium, player 2 plays  $l$  after  $L$ , which is not a best response in the original game. Therefore, one may suspect that although the contract specified in the previous paragraph is optimal, it may be weakly dominated by an alternative contract that leads to best response behavior, i.e., playing  $r$ , after both  $L$  and  $R$ . Consider such a contract, say  $g$ , and note that incentive compatibility implies  $g(l) \geq g(r) - 1$  (otherwise player 2 would play  $l$  after  $L$ ). Furthermore, we need to have  $g(r) \geq \delta$ , for otherwise the third party would reject  $g$  and player 2 would obtain some small payoff. Therefore,  $g(r) \geq \delta = f(r)$  and  $g(l) \geq \delta - 1 = f(l)$ , and thus, for any outcome of the game, player 2's payoff after  $f$  is at least as large as her payoff after  $g$ , which shows that  $f$  is not weakly dominated by any contract.<sup>14</sup>

Second, we assumed that the third party accepts any contract that gives him an equilibrium payoff of at least  $\delta$ . In particular, we allowed the contract to pay him less than  $\delta$  under some, out-of-equilibrium, circumstances. One might find this unreasonable on the grounds that if player 1 or player 2 makes a mistake in the game that ensues, the third party may end up with a payoff that is smaller than  $\delta$ , and therefore he would reject such a contract. One way to address this concern is to model the individual rationality constraint of the third party so that he requires a payment of at least  $\delta$  for every action player 2 might take. This would not change the set of outcomes that can be supported by renegotiable contracts, but may affect how the equilibrium surplus is shared between player 2 and the third party. For example, the least costly such contract that supports the outcome  $(R, r)$  would be given by  $f(l) = \delta$  and  $f(r) = 1 + \delta$ , in which case the equilibrium payoff of player 2 would be  $2 - \delta$ , rather than  $3 - \delta$ .<sup>15</sup>

Third, in the equilibrium constructed above, the third party believes that player 1 has played  $R$  after any out-of-equilibrium renegotiation offer. This might be regarded unreasonable, for there may be contracts that are suboptimal for player 2 to offer after  $R$ , but not after  $L$ . Therefore, one might want to restrict beliefs to  $L$  after such offers. This would be nothing but an application of the intuitive criterion (Cho and Kreps(1987)). It is easy to show that the outcome  $(R, r)$  can also be supported in an equilibrium that satisfies the intuitive criterion. More generally, in section 7 we show that all our results go through with minor modifications if we were to adopt this stronger notion of equilibrium.

<sup>13</sup>This has been first observed by Katz (1991) for the ultimatum bargaining game.

<sup>14</sup>We should also note that the outcome supported by  $f$ , i.e.,  $(R, r)$ , is a trembling hand perfect equilibrium outcome (see also Fershtman and Kalai (1997)).

<sup>15</sup>Another way would be to transfer some of the bargaining power to the third party so that in equilibrium he receives more than  $\delta$ . If, for example,  $\delta \leq 0.5$ , then  $f(r) = 1.5$  and  $f(l) = 0.5$  would support the same outcome in a way that always gives the third party at least  $\delta$ .

## SEQUENTIAL BATTLE-OF-THE-SEXES

Consider now the sequential battle-of-the-sexes game given in Figure 2. This game also has a unique SPE, given by  $(L, lr)$  and another Nash equilibrium,  $(R, rr)$ . It can be shown easily that the unique equilibrium outcome of the induced game with third-party contracts is  $(R, r)$  if the contracts are observable, non-renegotiable, and complete, whereas the SPE outcome  $(L, l)$  can also be supported if contracts are unobservable. If contracts are complete and renegotiable, then only the SPE outcome can be supported. All these observations are in line with those made for the ultimatum bargaining game.

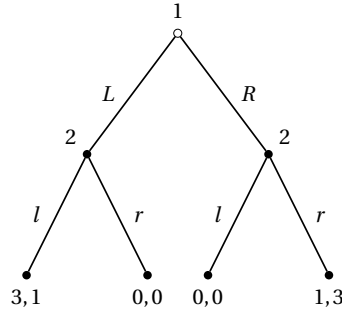


Figure 2: Battle-of-the-Sexes Game

However, the conclusion differs drastically from that in the ultimatum bargaining example if we assume that player 1's action is unobservable to the third party and contracts are renegotiable and incomplete. In this game only the SPE outcome can be supported, while in ultimatum bargaining a non-SPE outcome could also be supported.

Let us prove that the Nash equilibrium outcome  $(R, r)$  cannot be supported by renegotiable contracts. Suppose, for contradiction, that there exists a PBE of the game that supports this outcome. Let  $f : \{l, r\} \rightarrow \mathbb{R}$  be the equilibrium contract that specifies the transfer to be made from player 2 to the third party. For this outcome to be supported, player 2 must be playing  $r$  after both actions. Now consider the renegotiation offer by player 2 given by  $g(l) = g(r) = f(r) + 0.5$  after player 1 plays  $L$ . Note that the third party does not know which action has been played by player 1 when faced with this renegotiation offer. If he accepts  $g$ , he will receive a payoff of  $f(r) + 0.5$  irrespective of player 1's action. If, on the other hand, he rejects it, he believes that player 2 will play  $r$  after any action by player 1 and hence he will receive a payoff of  $f(r)$ .<sup>16</sup> Therefore, whatever his beliefs are regarding player 1's action, he has an incentive to accept this renegotiation offer. Furthermore, player 2 has an incentive to make such an offer after player 1 plays  $L$  since under  $f$  her expected payoff is  $-f(r)$ , whereas under  $g$  her expected payoff is  $-f(r) + 0.5$ . This establishes that there is no PBE that supports the outcome  $(R, r)$  with renegotiable contracts. Indeed, the unique outcome that can be supported in this case is the SPE outcome of the original game, i.e.,  $(L, l)$ .

In this section we presented two games that are superficially similar but for which renegotiable

<sup>16</sup>Here, and in the rest of the paper, we restrict player 2's strategy in the game to remain the same if her renegotiation offer is rejected. This is what allows us to state that the third party believes player 2 will play  $r$  if he rejects  $g$ . Otherwise, outcome  $(R, r)$  could also be supported in equilibrium, which would involve a change in player 2's behavior as a result of a failed attempt to change the contract. Since one of our objectives is to identify conditions under which non-subgame perfect outcomes can be supported by renegotiation-proof contracts, we disregard equilibria in which this happens as a result of arbitrary changes in behavior that the mere possibility of renegotiation introduces.

third-party contracts lead to completely different outcomes. In the rest of the paper we will provide an answer as to why this is the case and characterize outcomes that can be supported with renegotiable contracts in arbitrary two-stage extensive form games.

## 5 Main Results

In our model, if an outcome can be supported with renegotiable contracts, it can also be supported with non-renegotiable contracts.<sup>17</sup> Therefore, we start by characterizing the set of outcomes that can be supported with non-renegotiable contracts before we analyze the restrictions imposed by renegotiation. We should emphasize that  $\Gamma(G)$  is with unobservable but *incomplete* contracts. The results provided in Koçkesen and Ok (2004) are valid only for games with *complete contracts* and hence do not provide the relevant starting point for our analysis. Applied to our setting, Koçkesen and Ok (2004) implies that every Nash equilibrium outcome can be supported with complete contracts, whereas, as we will see in the next section, only a subset of these can be supported when contracts are incomplete.

### 5.1 Non-Renegotiable Contracts

Let  $G$  be an arbitrary original game and  $\Gamma(G)$  be the game with incomplete and non-renegotiable third-party contracts. We first prove the following.

**Proposition 1.** *A strategy profile  $(b_1^*, b_2^*)$  of  $G$  can be supported with incomplete and non-renegotiable contracts if and only if*

1.  $(b_1^*, b_2^*)$  is a Nash equilibrium of  $G$   
and there exists an  $f \in \mathcal{C}$  such that
2.  $f(b_2^*(b_1^*)) = \delta$ ,
3.  $u_2(a_1, b_2^*(a_1)) - f(b_2^*(a_1)) \geq u_2(a_1, b_2^*(a_1')) - f(b_2^*(a_1'))$ , for all  $a_1, a_1' \in A_1$ .

Proposition 1 provides necessary and sufficient conditions for an outcome of an arbitrary original game to be supported with incomplete and non-renegotiable contracts. Condition 1 states that only Nash equilibrium outcomes can be supported, which, as in Koçkesen and Ok (2004), follows from unobservability of contracts and sequential rationality of players 1 and 2. Condition 2 simply states that the third party does not receive rents in equilibrium, whereas condition 3 is the incentive compatibility constraint imposed by the incompleteness of contracts.

Although Proposition 1 provides a complete characterization, it falls short of precisely identifying the supportable outcomes in terms of the primitives of the original game. As is standard in adverse selection models, we can obtain a much sharper characterization if we impose an order structure on  $A_1$  and  $A_2$  and assume that player 2's payoff function  $v_2$  exhibits increasing differences. Given the definition of  $v_2$ , this is equivalent to assuming that  $u_2$  has increasing differences. To this end, let  $\succsim_1$  be a linear order on  $A_1$  and  $\succsim_2$  a linear order on  $A_2$ , and denote their asymmetric parts by  $\succ_1$  and  $\succ_2$ , respectively.

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<sup>17</sup>This claim is proved as (the [Only if]) part of Proposition 2 in Section 9.

**Definition 3** (Increasing Differences).  $u_2 : A_1 \times A_2 \rightarrow \mathbb{R}$  is said to have *increasing differences* in  $(\succsim_1, \succsim_2)$  if  $a_1 \succsim_1 a'_1$  and  $a_2 \succsim_2 a'_2$  imply that  $u_2(a_1, a_2) - u_2(a_1, a'_2) \geq u_2(a'_1, a_2) - u_2(a'_1, a'_2)$ . It is said to have *strictly increasing differences* if  $a_1 \succ_1 a'_1$  and  $a_2 \succ_2 a'_2$  imply that  $u_2(a_1, a_2) - u_2(a_1, a'_2) > u_2(a'_1, a_2) - u_2(a'_1, a'_2)$ .

**Definition 4** (Increasing Strategies).  $b_2 : A_1 \rightarrow A_2$  is called *increasing* in  $(\succsim_1, \succsim_2)$  if  $a_1 \succsim_1 a'_1$  implies that  $b_2(a_1) \succsim_2 b_2(a'_1)$ .

From now on, we restrict our analysis to games in which there exists a linear order  $\succsim_1$  on  $A_1$  and a linear order  $\succsim_2$  on  $A_2$  such that  $u_2$  has strictly increasing differences in  $(\succsim_1, \succsim_2)$ . We then have the following result.

**Theorem 1.** *A strategy profile  $(b_1^*, b_2^*)$  of  $G$  can be supported with incomplete and non-renegotiable contracts if and only if  $(b_1^*, b_2^*)$  is a Nash equilibrium of  $G$  and  $b_2^*$  is increasing.*

This result completely characterizes the strategy profiles that can be supported with incomplete contracts and precisely identifies the restrictions imposed by incompleteness. While earlier papers showed that any Nash equilibrium of the original game can be supported by unobservable and *complete* contracts, this result shows that only the subset of Nash equilibria in which the second player plays an increasing strategy can be supported if, instead, contracts are incomplete.

The reason why only increasing strategies of the second player can be supported is very similar to the reason why only increasing strategies can be supported in standard adverse selection models: If the payoff function of player 2 exhibits increasing differences, then incentive compatibility is equivalent to increasing strategies. The set of actions of player 1,  $A_1$ , plays the role of the type set of the agent in standard principal-agent models. The fact that contracts cannot be conditioned on  $A_1$  transforms the model into an adverse selection model, which, combined with increasing differences exhibited by  $u_2(a_1, a_2) - f(a_2)$ , necessitates increasing strategies to satisfy incentive compatibility, i.e., condition 3 of Proposition 1. We prove sufficiency by using a theorem of the alternative.

## 5.2 Renegotiable Contracts

Let  $G$  be an arbitrary original game and  $\Gamma_R(G)$  be the induced game with incomplete and renegotiable third-party contracts. As stated before we would like to identify the set of outcomes of  $G$  that can be supported by renegotiation-proof perfect Bayesian equilibria of  $\Gamma_R(G)$ .

When faced with a renegotiation offer, the third party has to form beliefs regarding how player 2 would play under the new contract and compare his payoffs from the old and the new contracts to decide whether to accept it or not. As we have seen in section 5.1, contract incompleteness imposes incentive compatibility constraints on the strategy of player 2, and therefore the third party has to restrict his beliefs to strategies that are incentive compatible under the new contract. For future reference, let us first define incentive compatibility as a property of any contract-strategy pair  $(f, b_2) \in \mathcal{C} \times A_2^{A_1}$ .

**Definition 5** (Incentive Compatibility).  $(f, b_2) \in \mathcal{C} \times A_2^{A_1}$  is *incentive compatible* if

$$u_2(a_1, b_2(a_1)) - f(b_2(a_1)) \geq u_2(a_1, b_2(a'_1)) - f(b_2(a'_1)) \text{ for all } a_1, a'_1 \in A_1.$$

To understand the constraints imposed by renegotiation-proofness suppose that  $(\beta, \mu)$  is a renegotiation-proof PBE of  $\Gamma_R(G)$  and let  $f$  be the equilibrium contract and  $b_2^*$  be the equilibrium strategy of player 2 following  $f$ . Now suppose that for a particular choice of action by player 1, say  $a'_1$ , there exists an incentive compatible contract-strategy pair  $(g, b_2)$  such that  $u_2(a'_1, b_2(a'_1)) - g(b_2(a'_1)) > u_2(a'_1, b_2^*(a'_1)) - f(b_2^*(a'_1))$  and  $g(b_2(a_1)) > f(b_2^*(a_1))$  for all  $a_1$ . This implies that, after  $a'_1$  is played, player 2 will have an incentive to renegotiate and offer  $g$  and the third party will have an incentive to accept it. This would contradict that  $(\beta, \mu)$  is a renegotiation-proof PBE of  $\Gamma_R(G)$ . This leads to the following definition.

**Definition 6** (Renegotiation-Proofness). We say that  $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1}$  is *renegotiation-proof* if for all  $a_1 \in A_1$  for which there exists an incentive compatible  $(g, b_2) \in \mathcal{C} \times A_2^{A_1}$  such that

$$u_2(a_1, b_2(a_1)) - g(b_2(a_1)) > u_2(a_1, b_2^*(a_1)) - f(b_2^*(a_1)) \quad (1)$$

and

$$g(b_2(a_1)) > f(b_2^*(a_1)) \quad (2)$$

there exists an  $a'_1 \in A_1$  such that

$$f(b_2^*(a'_1)) \geq g(b_2(a'_1)) \quad (3)$$

Again, the intuition behind this definition is clear: Whenever there is an  $a_1$  after which there is a contract  $g$  and an incentive compatible continuation play  $b_2$  such that the contracting parties both prefer  $g$  over  $f$  (i.e., (1) and (2) hold), there exists a belief of the third party under which it is optimal to reject  $g$ , which is implied by (3).<sup>18</sup>

In a similar vein, we have the following definition for a renegotiation-proof strategy.

**Definition 7** (Renegotiation-Proof Strategy). A strategy  $b_2 \in A_2^{A_1}$  is *renegotiation-proof* if there exists an  $f \in \mathcal{C}$  such that  $(f, b_2)$  is incentive compatible and renegotiation-proof.

Intuitively, Definition 7 seems to identify the conditions that  $b_2$  must satisfy to be induced by a renegotiation-proof perfect Bayesian equilibrium of  $\Gamma_R(G)$ . The following result proves that this intuition is correct.

**Proposition 2.** *A strategy profile  $(b_1^*, b_2^*)$  of  $G$  can be supported with incomplete and renegotiable contracts if and only if  $(b_1^*, b_2^*)$  is a Nash equilibrium of  $G$  and  $b_2^*$  is increasing and renegotiation-proof.*

Unfortunately, it is difficult to apply Definitions 6 and 7 directly to an arbitrary game to ascertain the restrictions that renegotiation-proofness imposes on contracts and strategies. However, the conditions themselves are all linear inequalities and we can use theorems of the alternative to understand these restrictions better in terms of the primitives of the original game. To this end, let the number of elements in  $A_1$  be equal to  $n$  and order its elements so that  $a_1^n \succsim_1 a_1^{n-1} \succsim_1 \dots \succsim_1 a_1^2 \succsim_1 a_1^1$ . For any contract-strategy pair  $(f, b_2)$ , define  $f_j = f(b_2(a_1^j))$ ,  $j = 1, \dots, n$ , and let, with an abuse of notation,  $f \in \mathbb{R}^n$  be the vector whose  $j$ th component is given by  $f_j$ .

First, note that, under increasing differences, incentive compatibility of  $(g, b_2)$  is equivalent to  $b_2$  being increasing and conditions (1) and (2) imply that  $u_2(a_1, b_2(a_1)) > u_2(a_1, b_2^*(a_1))$ . In other words,

<sup>18</sup>One may find this definition too weak as it allows the beliefs to be arbitrary following an off-the-equilibrium renegotiation offer. A more reasonable alternative could be to require the beliefs to satisfy intuitive criterion. In Section 7 we show that our results go through with minor modifications when we adopt this stronger version of renegotiation-proofness.

condition (3) needs to be satisfied for every  $a_1^i, i = 1, \dots, n$ , and increasing strategy that leads to a higher surplus for the contracting parties. For a given  $i$ , let us define the set of all such strategies as

$$\mathfrak{B}(i, b_2^*) = \{b_2 \in A_2^{A_1} : b_2 \text{ is increasing and } u_2(a_1^i, b_2(a_1^i)) > u_2(a_1^i, b_2^*(a_1^i))\}. \quad (4)$$

Second, by Definition 6,  $(f, b_2^*)$  is *not* renegotiation-proof if and only if there exist  $i$  and incentive compatible  $(g, b_2)$  such that  $u_2(a_1^i, b_2(a_1^i)) - g_i > u_2(a_1^i, b_2^*(a_1^i)) - f_i$  and  $g_j > f_j$  for all  $j$ . When  $u_2$  has increasing differences, incentive compatibility of  $(g, b_2)$  is equivalent to the local upward and downward constraints:

$$\begin{aligned} g_j - g_{j+1} &\leq u_2(a_1^j, b_2(a_1^j)) - u_2(a_1^j, b_2(a_1^{j+1})), \quad j = 1, \dots, n-1 \\ -g_{j-1} + g_j &\leq u_2(a_1^j, b_2(a_1^j)) - u_2(a_1^j, b_2(a_1^{j-1})), \quad j = 2, \dots, n \end{aligned}$$

We can write these inequalities in matrix form as  $Dg \leq U(b_2)$ , where  $D$  is a matrix of coefficients and  $U(b_2)$  a column vector with  $2(n-1)$  components, whose component  $2j-1$  is given by

$$U(b_2)_{2j-1} = u_2(a_1^j, b_2(a_1^j)) - u_2(a_1^j, b_2(a_1^{j+1}))$$

and component  $2j$  is given by

$$U(b_2)_{2j} = u_2(a_1^{j+1}, b_2(a_1^{j+1})) - u_2(a_1^{j+1}, b_2(a_1^j))$$

Therefore,  $(f, b_2^*)$  is not renegotiation-proof if and only if there exist  $i, b_2$ , and  $\varepsilon \in \mathbb{R}^n$  such that

$$D(f + \varepsilon) \leq U(b_2), \quad \varepsilon_i < u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)), \quad \varepsilon \gg 0$$

These conditions can be written as  $[Ax \gg 0, Cx \geq 0 \text{ has a solution } x]$ , once the vector  $x$  and matrices  $A$  and  $C$  are appropriately defined. Motzkin's theorem of the alternative (stated as Lemma 3 in section 9) then implies that the necessary and sufficient condition for being renegotiation-proof is  $[A'y_1 + C'y_2 = 0, y_1 > 0, y_2 \geq 0 \text{ has a solution } y_1, y_2]$  (See Lemma 4 in section 9). The fact that  $u_2$  has increasing differences can then be used to prove the equivalence of this condition to the one stated in the following theorem.

**Theorem 2.**  $(f, b_2^*)$  is renegotiation-proof if and only if for any  $i \in \{1, 2, \dots, n\}$  and  $b_2 \in \mathfrak{B}(i, b_2^*)$  there exists a  $k \in \{1, 2, \dots, i-1\}$  such that

$$u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \sum_{j=k}^{i-1} U(b_2)_{2j-1} \leq f_k - f_i \quad (5)$$

or there exists an  $l \in \{i+1, i+2, \dots, n\}$  such that

$$u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \sum_{j=i+1}^l U(b_2)_{2(j-1)} \leq f_l - f_i \quad (6)$$

In order to apply this theorem directly to a given game and a strategy  $b_2^*$  one would first identify the set of contracts under which player 2 has an incentive to play  $b_2^*$ , and then check if any of those contracts satisfies the conditions of the theorem. It is best to illustrate this using the examples in-

troduced in Section 4. For both the ultimatum bargaining and sequential battle-of-the-sexes games, define  $\succsim_1$  and  $\succsim_2$  so that  $R \succ_1 L$  and  $r \succ_2 l$  and note that  $u_2$  has strictly increasing differences in  $(\succsim_1, \succsim_2)$ .

#### ULTIMATUM BARGAINING

There are three Nash equilibria of the game:  $(L, rl)$ ,  $(L, rr)$ , and  $(R, lr)$ . The second one is the unique SPE and it has the same outcome as the first. The third one is not subgame perfect. Notice that the last two equilibria have increasing  $b_2$  and hence, by Theorem 1, can be supported with incomplete and non-renegotiable contracts. Since the SPE can be supported with renegotiable contracts as well, the question is whether  $(R, lr)$  can be supported with incomplete and renegotiable contracts.<sup>19</sup>

Any equilibrium contract  $f$  that supports  $(R, lr)$  must satisfy the incentive compatibility constraint given by  $1 \leq f(r) - f(l) \leq 3$ . Since player 2 is already best responding after  $R$ , a surplus-increasing renegotiation can happen only after  $L$  and it must lead to  $b_2(L) = r$ . Incentive compatibility implies that  $b_2$  is increasing, and therefore,  $b_2(R) = r$ . From Theorem 2,  $(f, b_2^*)$  is renegotiation-proof if and only if

$$[u_2(L, b_2(L)) - u_2(L, b_2^*(L))] + [u_2(R, b_2(R)) - u_2(R, b_2(L))] \leq f(b_2^*(R)) - f(b_2^*(L))$$

Substituting for  $b_2^*$  and  $b_2$ , this is equivalent to  $1 \leq f(r) - f(l)$ . Since incentive compatibility holds if  $1 \leq f(r) - f(l) \leq 3$ , we conclude that  $b_2^* = lr$  can be supported with a renegotiation-proof contract and hence  $(R, lr)$  can be supported with incomplete and renegotiable contracts.

#### SEQUENTIAL BATTLE-OF-THE-SEXES

There are three Nash equilibria of the game:  $(L, ll)$ ,  $(L, lr)$ , and  $(R, rr)$ . The second one is the unique SPE and it has the same outcome as the first. The third one is not subgame perfect. All of these equilibria have an increasing  $b_2$  and hence can be supported with incomplete and non-renegotiable contracts. The question again is whether the (non-subgame perfect) Nash equilibrium  $(R, rr)$  can be supported with incomplete and renegotiable contracts.

The only possibility for a surplus-increasing renegotiation is after  $L$  and it must induce  $b_2(L) = l$ . Theorem 2 implies that if  $(f, b_2^*)$  is renegotiation-proof then

$$[u_2(L, b_2(L)) - u_2(L, b_2^*(L))] + [u_2(R, b_2(R)) - u_2(R, b_2(L))] \leq f(b_2^*(R)) - f(b_2^*(L))$$

or  $u_2(R, b_2(R)) + 1 \leq 0$ , which is impossible since  $u_2(R, b_2(R)) \geq 0$ . We conclude that it is not possible to support  $(R, rr)$  with incomplete and renegotiable contracts.

Although Theorem 2 is quite powerful in applications, it would still be desirable to obtain general results that involve only the primitives of the original game. In particular, we would like to obtain conditions for a strategy  $b_2^*$  to be supportable with incomplete and renegotiable contracts. Given Proposition 2, this requires identifying renegotiation-proof strategies, i.e., those for which there exists an  $f \in \mathcal{C}$  such that  $(f, b_2^*)$  is incentive compatible and renegotiation-proof. For any  $k, i \in \{1, \dots, n\}$ , incentive compatibility implies

$$f_k - f_i \leq u_2(a_1^k, b_2^*(a_1^k)) - u_2(a_1^i, b_2^*(a_1^i)).$$

<sup>19</sup>Clearly, if a contract supports a SPE, it is renegotiation-proof as there is no  $a_1 \in A_1$  such that (1) and (2) hold.

Together with Theorem 2, we then have the following necessary condition for  $b_2^*$  being renegotiation-proof: For any  $i = 1, \dots, n$  and  $b_2^i \in \mathfrak{B}(i, b_2^*)$  there exists a  $k \in \{1, 2, \dots, i-1\}$  such that

$$u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \sum_{j=k}^{i-1} U(b_2)_{2j-1} \leq u_2(a_1^k, b_2^*(a_1^k)) - u_2(a_1^i, b_2^*(a_1^i))$$

or there exists an  $l \in \{i+1, i+2, \dots, n\}$  such that

$$u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \sum_{j=i+1}^l U(b_2)_{2(j-1)} \leq u_2(a_1^l, b_2^*(a_1^l)) - u_2(a_1^i, b_2^*(a_1^i))$$

In fact, again utilizing a theorem of the alternative (Gale's theorem of inequalities), we can make this condition tighter. To facilitate the exposition, we first introduce the following definition.

**Definition 8.** For any  $i = 1, \dots, n$  and  $b_2 \in \mathfrak{B}(i, b_2^*)$  we say that  $m(b_2) \in \{1, 2, \dots, n\}$  is a *blocking action* if

$$u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \sum_{j=m(b_2)}^{i-1} U(b_2)_{2j-1} \leq \sum_{j=m(b_2)}^{i-1} U(b_2^*)_{2j-1} \quad (7)$$

or

$$u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \sum_{j=i+1}^{m(b_2)} U(b_2)_{2(j-1)} \leq \sum_{j=i+1}^{m(b_2)} U(b_2^*)_{2(j-1)} \quad (8)$$

We then obtain the following result.

**Proposition 3.** A strategy  $b_2^* \in A_2^{A_1}$  is renegotiation-proof only if for any  $i = 1, \dots, n$  and  $b_2^i \in \mathfrak{B}(i, b_2^*)$  there is a blocking action.<sup>20</sup>

However, this condition is not sufficient for renegotiation-proofness and becomes sufficient with an additional condition on the blocking actions for different  $a_1$ 's. More precisely,

**Proposition 4.** A strategy  $b_2^* \in A_2^{A_1}$  is renegotiation-proof if for any  $i = 1, \dots, n$  and  $b_2^i \in \mathfrak{B}(i, b_2^*)$  there is a blocking action  $m(b_2^i)$  such that  $k < l$ ,  $m(b_2^k) > k$ , and  $m(b_2^l) < l$  imply  $m(b_2^k) \leq m(b_2^l)$ .

The conditions given in Propositions 3 and 4 coincide when player 1 has only two actions. Therefore, Proposition 3 is a full characterization result for such games, including our running examples. Let us use this proposition to show that  $b_2^* = lr$  is renegotiation-proof in ultimatum bargaining example. Let  $L = a_1^1$  and  $R = a_1^2$  and note that  $\mathfrak{B}(1, b_2^*) = \{rr\}$  and  $\mathfrak{B}(2, b_2^*) = \emptyset$ . Therefore, we only need to check if there is a blocking action for  $i = 1$  and  $b_2 = rr$ , the only candidate for which is  $R$ . Applying (8), we get

$$u_2(L, r) - u_2(L, l) + u_2(R, r) - u_2(R, r) \leq u_2(R, r) - u_2(R, l)$$

which is satisfied. We therefore conclude that  $b_2^* = lr$  is renegotiation proof.

Now let us show that  $b_2^* = rr$  is not renegotiation-proof in the battle-of-the-sexes game. In this case,  $\mathfrak{B}(2, b_2^*) = \emptyset$  and  $\mathfrak{B}(1, b_2^*) = \{ll, lr\}$ . It is sufficient to show that there is no blocking action for  $i = 1$  and  $b_2 = lr$ . The only candidate for a blocking action is  $R$  and we need the following inequality to be satisfied

$$u_2(L, l) - u_2(L, r) + u_2(R, r) - u_2(R, l) \leq u_2(R, r) - u_2(R, r).$$

<sup>20</sup>The fact that this is a tighter condition follows from an easy induction argument that establishes  $\sum_{j=k}^{i-1} U(b_2^*)_{2j-1} \leq u_2(a_1^k, b_2^*(a_1^k)) - u_2(a_1^i, b_2^*(a_1^i))$ .

Obviously, this is not true and we conclude that  $b_2^* = rr$  is not renegotiation-proof.

When  $A_1$  has more than two actions the condition stated in Proposition 3 is not sufficient anymore and obtaining a full characterization for such games requires introducing more structure into the model. In the next section we do this for a large class of economically relevant games.

## 6 A Special Environment and Applications

In this section we analyze a class of games that includes many economic models, among which are Stackelberg and entry games, sequential Bertrand games with differentiated products, relationship specific investment, and ultimatum bargaining. To define this class of games, take any original game  $G$  and consider the strategic form game  $S(G) = (\{1, 2\}, (A_1, A_2), (u_1, u_2))$ , i.e.,  $S(G)$  is the simultaneous move version of  $G$ . Let  $br_i$  denote a selection from the best-response correspondence of player  $i$  in  $S(G)$ , i.e.,  $br_i(a_{-i}) \in BR_i(a_{-i})$  for all  $a_{-i} \in A_{-i}$ . Also, let  $NE(S(G))$  be the set of pure strategy Nash equilibria of  $S(G)$  and denote the smallest and the largest (in  $\succsim_1$ ) pure strategy Nash equilibrium action of player 1 by  $\underline{a}_1^{NE}$  and  $\bar{a}_1^{NE}$ , respectively.

**Definition 9.**  $u_1$  has *positive externality* in  $\succsim_2$  if  $a_2 \succsim_2 a'_2$  implies  $u_1(a_1, a_2) \geq u_1(a_1, a'_2)$  for all  $a_1 \in A_1$ .

**Definition 10.**  $u_1$  is *single-peaked* in  $\succsim_1$  if for all  $br_1 \in BR_1$  and  $a_2 \in A_2$ ,  $br_1(a_2) \succsim_1 a'_1 \succsim_1 a_1$  implies  $u_1(a'_1, a_2) \geq u_1(a_1, a_2)$  and  $a_1 \succsim_1 a'_1 \succsim_1 br_1(a_2)$  implies  $u_1(a'_1, a_2) \geq u_1(a_1, a_2)$ . Define single-peaked  $u_2$  in a similar manner.

**Definition 11.** For any original game  $G$ , we say that  $NE(S(G))$  is *stable* if, for any selection  $(br_1, br_2)$ ,  $a_1 \succsim_1 \bar{a}_1^{NE}$  implies  $a_1 \succsim_1 br_1(br_2(a_1))$  and  $a_1 \precsim_1 \underline{a}_1^{NE}$  implies  $a_1 \precsim_1 br_1(br_2(a_1))$ .

Positive externality and single-peakedness are standard conditions. Stability condition, on the other hand, is necessary for the convergence of best response dynamics to the interval  $[\underline{a}_1^{NE}, \bar{a}_1^{NE}]$ . Indeed, if there exist  $a_1^0 \succ_1 \bar{a}_1^{NE}$  and  $(br_1, br_2)$  such that  $a_1^0 \prec_1 br_1(br_2(a_1^0))$ , then starting from  $(a_1^0, br_2(a_1^0))$ , the dynamics defined by  $(a_1^{t+1}, a_2^{t+1}) = (br_1(a_2^t), br_2(a_1^t))$ ,  $t = 0, 1, \dots$ , would be such that  $a_1^t \succ_1 \bar{a}_1^{NE}$  for all  $t = 0, 1, \dots$

Let  $\mathcal{G}$  denote the class of games  $G$  in which  $u_1$  and  $u_2$  are single-peaked,  $u_2$  has strictly increasing differences in  $(\succsim_1, \succsim_2)$ ,  $u_1$  has positive externality, and  $NE(S(G))$  is stable and non-empty.<sup>21</sup> Also, let  $\bar{a}_i = \max_{\succsim_i} A_i$  and  $\underline{a}_i = \min_{\succsim_i} A_i$ . The following result provides necessary and sufficient conditions for an outcome to be supported with incomplete and renegotiable contracts.

**Theorem 3.** Let  $G \in \mathcal{G}$ . An outcome  $(a_1^*, a_2^*)$  of  $G$  can be supported with incomplete and renegotiable contracts if (only if, resp.)  $a_1^* \succsim_1 \bar{a}_1^{NE}$  ( $a_1^* \precsim_1 \underline{a}_1^{NE}$ , resp.), and  $a_2^* = br_2(a_1^*)$ ,

$$u_1(a_1^*, br_2(a_1^*)) \geq \max \{u_1(br_1(\underline{a}_2), \underline{a}_2), u_1(\bar{a}_1, br_2(\bar{a}_1))\} \quad (9)$$

for some selection  $(br_1, br_2) \in BR_1 \times BR_2$ .

In other words, in this environment outcomes in which player 1 plays an action that is “smaller” than his smallest Nash equilibrium action (in the simultaneous move version of the original game) cannot be supported. Conversely, any outcome in which player 1’s action is greater than his largest

<sup>21</sup>Clearly, if  $u_2$  has strictly *decreasing* differences and  $u_1$  has *negative* externality, the game is still a member of  $\mathcal{G}$ .

Nash equilibrium action can be supported, as long as player 2 best responds to that action in a way that condition (9) is satisfied.

If  $S(G)$  has a unique Nash equilibrium, then Theorem 3 provides a full characterization and therefore a complete answer to our main query: An outcome of the original game  $G$  can be supported with incomplete and renegotiable contracts if and only if player 1's action is greater than his Nash equilibrium action in  $S(G)$ , player 2's action is a best response to that, and player 1's payoff is not smaller than his minmax value.<sup>22</sup>

One might want to identify the additional restrictions imposed by renegotiation on the set of outcomes that can be supported with third-party contracts. More precisely, what are the outcomes that can be supported with incomplete and *non-renegotiable* contracts, but not with incomplete and *renegotiable* contracts? A comparison between Theorem 1 and Theorem 2 does not provide a direct answer to this question, as the latter only characterizes the additional restrictions brought about by renegotiation on contract-strategy pairs, not outcomes. When we restrict our attention to games in  $\mathcal{G}$ , the following corollary of Theorem 3 provides a complete answer and identifies an environment in which renegotiation has no bite at all.

**Corollary 1.** *Let  $G \in \mathcal{G}$ , and suppose that  $S(G)$  has a unique pure strategy Nash equilibrium, and that  $a_1 \succsim_1 a_1^{NE}$  implies  $u_1(a_1, br_2(a_1)) \geq u_1(\bar{a}_1, br_2(\bar{a}_1))$  for all  $br_2 \in BR_2$ . If an outcome  $(a_1^*, a_2^*)$  of  $G$  can be supported with incomplete and non-renegotiable contracts, then it can also be supported with incomplete and renegotiable contracts.*

It could be useful to summarize our main results so far:

1. If third-party contracts are incomplete and non-renegotiable, then the outcomes of the original game that can be supported are the subset of Nash equilibrium outcomes in which player 2 plays an increasing strategy (Theorem 1).
2. If we restrict attention to games in  $\mathcal{G}$  whose simultaneous move version has a unique pure strategy Nash equilibrium, all the outcomes that can be supported with incomplete and non-renegotiable contracts can also be supported with incomplete and renegotiable contracts. These outcomes are those in which player 1 chooses an action greater than his Nash equilibrium action and player 2 best responds (Theorem 3 and Corollary 1).
3. In games that do not belong to  $\mathcal{G}$ , there may be outcomes that can be supported with incomplete and non-renegotiable contracts, but not with renegotiable contracts (see the sequential battle-of-the-sexes example). For these environments, Theorem 2 provides a characterization in terms of renegotiation-proof contract-strategy pairs.

Finally, we should note that if  $u_1$  has strictly increasing differences as well as  $u_2$ , then  $S(G)$  is a supermodular game and hence it has a smallest and largest pure strategy Nash equilibria (Topkis (1979)). Furthermore, it can be shown that  $NE(S(G))$  is stable in the sense of Definition 11.<sup>23</sup> Therefore, if  $G$  is such that  $u_1$  and  $u_2$  have strictly increasing differences, are single-peaked, and  $u_1$  has positive externality, then  $G \in \mathcal{G}$ .

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<sup>22</sup>Since we assumed that  $u_1$  has positive externalities in  $\succsim_2$ , the minmax value for player 1 is  $\min_{a_2} \max_{a_1} u_1(a_1, a_2) = u_1(br_1(\underline{a}_2), \underline{a}_2)$ . In many games, the second component of the condition (9) is satisfied trivially.

<sup>23</sup>This assertion is proved in Section 9 as Lemma 10.

## 6.1 Applications

In this section, we illustrate the results of our paper using three simple models. The first two models study the impact of third-party contracts in oligopolistic competition and allow us to compare our results with those in Caillaud et al. (1995) and Dewatripont (1988). The last one is the ultimatum bargaining game. In all the models, renegotiable third-party contracts can be used as a commitment device, but the degree to which player 2 can benefit from this depends on the game.

It is important to notice that all the games in this section belong to the class  $\mathcal{G}$  and satisfy Corollary 1. Therefore, these are all situations in which renegotiation has no bite.

### APPLICATION I: QUANTITY COMPETITION AND ENTRY-DETERRENCE

Consider a Stackelberg game in which firm 1 moves first by choosing an output level  $q_1 \in Q_1$  and firm 2, after observing  $q_1$ , chooses its own output level  $q_2 \in Q_2$ . We assume that  $Q_i$ ,  $i = 1, 2$ , is a rich enough finite subset of  $\mathbb{R}_+$  and includes 0 and 12.<sup>24</sup> Inverse demand function is given by  $p(q_1, q_2) = \max\{0, 12 - q_1 - q_2\}$  and, for simplicity, cost function is set to zero. Therefore, profit function of firm  $i$  is given by  $\pi_i(q_1, q_2) = p(q_1, q_2)q_i$  and we assume that both firms are profit maximizers. The subgame perfect (Stackelberg) equilibrium outcome of this game is given by  $(q_1, q_2) = (6, 3)$ , with corresponding profits  $(\pi_1, \pi_2) = (18, 9)$ , while the (Cournot) Nash equilibrium of the simultaneous version of the same game is given by  $(q_1, q_2) = (4, 4)$ , with profits  $(\pi_1, \pi_2) = (16, 16)$ .

Define the game  $G$  as follows: Let  $A_1 = Q_1$  and  $A_2 = \{-q_2 : q_2 \in Q_2\}$  and define  $\succsim_i$  on  $A_i$  as  $a_i \succsim_i a'_i \Leftrightarrow a_i \geq a'_i$ . Let the payoff function of player  $i$  be given by  $u_i(a_1, a_2) = \pi_i(a_1, -a_2)$ , for any  $(a_1, a_2) \in A_1 \times A_2$ . The game  $G$  is strategically equivalent to the Stackelberg game defined in the previous paragraph, while the simultaneous move version of it,  $S(G)$ , is the standard Cournot model.

It is easy to show that  $G \in \mathcal{G}$ , and hence we can apply Theorem 3, which implies that an outcome can be supported with incomplete and renegotiable contracts if and only if firm 1's profit is non-negative, its output is at least as high as the Cournot Nash equilibrium output, and the follower's output is a best response to that. More precisely, an outcome  $(q_1^*, q_2^*)$  can be supported if and only if  $4 \leq q_1^* \leq 12$  and  $q_2^* = 6 - q_1^*/2$ .

In this game, therefore, firm 2 may benefit from third-party contracts, even when they are renegotiable. However, there is a limit to how much firm 2 may benefit: It cannot force firm 1 to produce less than the Cournot Nash equilibrium output. Therefore, the best equilibrium payoff of firm 2 is the Cournot Nash equilibrium profit of 16. This example suggests that when applied to a Cournot-Stackelberg setting, third party contracts can benefit the follower by undoing firm 1's first mover advantage.

We could also interpret the above scenario as an *entry game* where the potential entrant, firm 1, is free to choose any capacity level,  $q_1$  in entering, and  $q_1 = 0$  is interpreted as no-entry. Theorem 3 then implies that, in any equilibrium of the entry game with third-party contracts, the entrant chooses a positive capacity level, i.e., third-party contracts cannot deter entry, which stands in contrast to the result obtained in Dewatripont (1988). If, as in our paper, the asymmetry of information between the incumbent and the third party originates from the inability of the third party to observe the move of the entrant, then entry cannot be deterred by third party contracts. If, as in Dewatripont's model, the asymmetry of information is on some parameter (for example the payoff function of the third

<sup>24</sup>Including these two points allows us to calculate  $\max\{u_1(br_1(\underline{a}_2), \underline{a}_2), u_1(\bar{a}_1, br_2(\bar{a}_1))\} = 0$ .

party), but the move of the entrant is observed by both the incumbent and the third party, then the incumbent can deter entry.

#### APPLICATION II: PRICE COMPETITION

The second application is a simple model of price (Bertrand) competition between two firms that produce differentiated products. Firm 1 moves first by choosing a price  $p_1 \in P_1$  and firm 2, after observing  $p_1$ , chooses its own price  $p_2 \in P_2$ . We assume that for each firm  $i = 1, 2$ ,  $P_i$  is a rich enough finite subset of  $\mathbb{R}_+$  and includes 0 and 12.<sup>25</sup> Firm  $i$ 's demand function is  $q_i(p_i, p_j) = \max\{0, 4 - p_i + p_j\}$ ,  $i \neq j$ , and, for simplicity, its cost is zero. Therefore, firm  $i$ 's profit function is given by  $\pi_i(p_i, p_j) = q_i(p_i, p_j)p_i$  and we assume that both firms are profit maximizers. The subgame perfect equilibrium outcome of this game is given by  $(p_1, p_2) = (6, 5)$ , with corresponding profits  $(\pi_1, \pi_2) = (18, 25)$ , while the Nash equilibrium of the simultaneous version of the same game is given by  $(p_1, p_2) = (4, 4)$ , with profits  $(\pi_1, \pi_2) = (16, 16)$ .

After endowing the set of actions with natural order, we can easily show that this game belongs to  $\mathcal{G}$  and apply Theorem 3. We conclude that an outcome  $(p_1^*, p_2^*)$  can be supported with incomplete and renegotiable contracts if and only if  $p_1^* \geq 4$ , i.e., the price of firm 1 is at least as high as the Nash equilibrium price in the simultaneous version,  $p_2^* = 2 + p_1^*/2$ , i.e., the price of firm 2 is a best response, and  $\pi_1(p_1^*, p_2^*) \geq 4$ , i.e., firm 1's profits are not lower than  $\pi_1(br_1(0), 0)$ , which implies that  $p_1^* \leq 6 + 2\sqrt{7}$ .

Therefore, in this model too, firm 2 may benefit from third party contracts, even when they are renegotiable. However, there is again a limit to how much it can benefit: This is given by condition (9) that firm 1 must receive at least its minmax payoff of 4, which implies that the highest profit firm 2 can get is  $32 + 10\sqrt{7} \approx 58.5$ . In other words, when applied to a price competition setting, third party contracts can benefit the second mover by reducing the first mover's profits to its minmax level.

Our results differ from the ones in Caillaud et al. (1995): They show that while third party contracts have commitment value under Cournot competition, this is not the case under Bertrand competition. In contrast, we show that third party contracts can benefit firm 2 in both settings. In Caillaud et al., third-party contracts have a strategic effect because principals can credibly announce that the agent will receive a higher utility than her reservation utility. This, in turn, provides incentives to increase the level of output, which is profitable only in the Cournot case. In our model, third-party contracts can provide incentives in a more flexible fashion: If firm 1 does not play the equilibrium action, the contract provides incentives to punish firm 1, by playing the lowest action  $\underline{a}_2$ .

#### APPLICATION III: ULTIMATUM BARGAINING GAME

Consider an *ultimatum bargaining game* in which the set of possible offers is  $A_1 = \{1, 2, \dots, n\}$ , for some integer  $n > 1$ , and  $A_2 = \{Y, N\}$ . Let  $a_1 \succsim_1 a'_1$  if and only if  $a_1 \geq a'_1$  and  $Y \succ_2 N$ . Suppose that if the responder (player 2) accepts an offer  $a_1$ , i.e., chooses  $Y$ , then the proposer's (player 1) payoff is  $n - a_1$  and that of the responder is  $a_1$ , while if the responder rejects, i.e., chooses  $N$ , they both get zero payoff. The unique subgame perfect equilibrium outcome of this game, as well as the unique Nash equilibrium outcome of its simultaneous move version, is given by  $(1, Y)$ , in which player 1 receives a payoff of  $n - 1$ , whereas player 2 receives 1.

This game satisfies all the assumptions required for Theorem 3 and condition (9) is trivially satisfied. We conclude that every outcome  $(a_1, Y)$ ,  $a_1 \in A_1$ , can be supported with incomplete and renegotiable contracts. In particular, player 2 can obtain the entire surplus  $n$ .

<sup>25</sup>This implies that  $\max\{u_1(br_1(\underline{a}_2), \underline{a}_2), u_1(\bar{a}_1, br_2(\bar{a}_1))\} = 4$ .

## 7 Strong Renegotiation-Proofness

One may object to our definition of renegotiation-proof perfect Bayesian equilibrium on the basis that off-the-equilibrium beliefs during the renegotiation process are left free. In particular, after the initial contract  $f$  and faced with an (off-the-equilibrium) renegotiation offer  $g$ , our definition allows the beliefs of the third party to assign positive probability to any action  $a_1$ . This enables us to construct a PBE in the proof of Proposition 2 in which the initial contract  $f$  is not renegotiated as long as  $(f, b_2^*)$  is renegotiation-proof as defined in Definition 6. A plausible way to strengthen our definition of renegotiation-proof equilibrium is to require that it satisfies the intuitive criterion as defined by Cho and Kreps (1987). When applied to our setting, this criterion requires that beliefs put positive probability only on actions for which  $(g, b_2)$  is not equilibrium-dominated, i.e., only on those actions  $a_1'$  for which  $u_2(a_1', b_2(a_1')) - g(b_2(a_1')) \geq u_2(a_1', b_2^*(a_1')) - f(b_2^*(a_1'))$ . This leads to the following definition.

**Definition 12** (Strong Renegotiation Proofness). We say that  $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1}$  is *strongly renegotiation-proof* if for all  $a_1 \in A_1$  for which there exists an incentive compatible  $(g, b_2) \in \mathcal{C} \times A_2^{A_1}$  such that

$$u_2(a_1, b_2(a_1)) - g(b_2(a_1)) > u_2(a_1, b_2^*(a_1)) - f(b_2^*(a_1)) \quad (10)$$

and

$$g(b_2(a_1)) > f(b_2^*(a_1)) \quad (11)$$

there exists an  $a_1' \in A_1$  such that

$$f(b_2^*(a_1')) \geq g(b_2(a_1')) \quad (12)$$

and

$$u_2(a_1', b_2(a_1')) - g(b_2(a_1')) \geq u_2(a_1', b_2^*(a_1')) - f(b_2^*(a_1')) \quad (13)$$

When we work with this definition, Theorem 2 needs to be modified as follows.

**Theorem 4.**  $(f, b_2^*)$  is strongly renegotiation-proof if and only if for any  $i \in \{1, 2, \dots, n\}$  and  $b_2 \in \mathfrak{B}(i, b_2^*)$  there exists a  $k \in \{1, 2, \dots, i-1\}$  such that

$$u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \sum_{j=k}^{i-1} U(b_2)_{2j-1} - \min\{0, u_2(a_1^k, b_2(a_1^k)) - u_2(a_1^k, b_2^*(a_1^k))\} \leq f_k - f_i \quad (14)$$

or there exists an  $l \in \{i+1, i+2, \dots, n\}$  such that

$$u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \sum_{j=i+1}^l U(b_2)_{2(j-1)} - \min\{0, u_2(a_1^l, b_2(a_1^l)) - u_2(a_1^l, b_2^*(a_1^l))\} \leq f_l - f_i \quad (15)$$

Also, it is easy to show that Proposition 2 and Theorem 3 (as well as Corollary 1) go through with strongly renegotiation-proof contracts, whereas Propositions 3 and 4 go through with a minor modification similar to the one made in Theorem 4.

## 8 Concluding Remarks

In this paper we characterized outcomes that can be supported in two-stage games with incomplete and non-renegotiable as well as renegotiable third-party contracts. We have seen that incomplete-

ness of the contracts restricts the outcomes that can be supported, in a natural way, to Nash equilibria in which the second mover's strategy is increasing (Theorem 1). Renegotiation imposes further constraints on these outcomes (Theorem 2) that limit them to subgame perfect equilibrium outcomes in some games. Yet, there is a large class of games in which non-subgame perfect equilibrium outcomes can also be supported with renegotiable third-party contracts. In particular, in an environment common to many economic models, such as the Stackelberg and ultimatum bargaining games, any outcome in which player 1 plays an action that is larger than his Nash equilibrium action in the simultaneous move version of the game and player 2 plays a best response can be supported with incomplete and renegotiable contracts (Theorem 3). In fact in many such environments renegotiation has no bite at all (Corollary 1).

There are several directions along which the current work can be extended in interesting ways. The most obvious of them is to consider more general information structures and contract spaces. One interesting possibility is to assume that the third party can observe only an outcome in some arbitrary outcome space  $Q$  and that only  $Q$  is contractible. The model is closed by assuming that there is a function  $p : A_1 \times A_2 \rightarrow Q$  such that  $p(q|a_1, a_2)$  is the probability of outcome  $q$  when  $(a_1, a_2)$  is played in the game. This introduces moral hazard issues into the model and might change our results in non-trivial ways. Another extension along similar lines would be a model in which player 2 has some payoff relevant information that is not available to the third-party. This is closer to a standard adverse selection model but is embedded in a strategic environment.<sup>26</sup> Characterization of renegotiation-proof outcomes in either of these models is left for future work.

Throughout the analysis we assumed that the original game is a finite two-stage game in which the second mover's set of actions is the same after any choice by the first mover. This allowed us to formulate incentive compatibility and renegotiation-proofness as sets of linear inequalities, which were relatively easy to manipulate and apply theorems of the alternative. A more technical extension of our work would be to consider arbitrary extensive form games. However, adapting the methods we used in the proofs to arbitrary games is not straightforward and this extension is also left for future work.

One important aspect of our paper is its use of theorems of the alternative to characterize incentive compatibility and renegotiation-proofness. We believe that these methods have the potential to be useful for models other than games with third-party contracts, such as characterizing renegotiation-proof contracts in dynamic principal-agent models or in single-person dynamic decision making problems with time-inconsistent preferences.

## 9 Proofs

In the game with incomplete and non-renegotiable contracts  $\Gamma(G)$ , player 2 has an information set at the beginning of the game, which we identify with the null history  $\emptyset$ , and an information set for each  $(f, a_1) \in \mathcal{C} \times A_1$ . Player 1 has only one information set, given by  $\mathcal{C}$ , and player 3 has an information set for each  $f \in \mathcal{C}$ . In  $\Gamma_R(G)$ , player 2 has additional information sets corresponding to each history  $(f, a_1, g, y)$  and  $(f, a_1, g, n)$  and player 3 has an additional information set of each  $(f, g) \in \mathcal{C}^2$ , which we denote by  $I_3(f, g)$ .

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<sup>26</sup>As we mentioned before, Dewatripont (1988) analyzes an example of such a model and shows that contracts can have a commitment value even under renegotiation.

*Proof of Proposition 1.* **[If]** Let  $(b_1^*, b_2^*)$  be a Nash equilibrium of  $G$  and  $f'$  satisfy the conditions of the proposition. For any  $b_2 \in A_2^{A_1}$ , let  $b_2(A_1)$  be the image of  $A_1$  under  $b_2$  and define

$$f^*(a_2) = \begin{cases} f'(a_2), & \text{if } a_2 \in b_2^*(A_1) \\ \max_{a_1} \{u_2(a_1, a_2) - u_2(a_1, b_2^*(a_1)) + f'(b_2^*(a_1))\}, & \text{otherwise} \end{cases}$$

for any  $a_2 \in A_2$ , and

$$b_{2,f}^*(a_1) = \begin{cases} b_2^*(a_1), & f = f^* \\ \in \operatorname{argmax}_{a_2} u_2(a_1, a_2) - f(a_2), & f \neq f^* \end{cases}$$

for any  $f \in \mathcal{C}$  and  $a_1 \in A_1$ . Consider the assessment  $(\beta^*, \mu^*)$  of  $\Gamma(G)$ , where  $\beta_2^*[\emptyset] = f^*$ ,  $\beta_1^*[\mathcal{C}] = b_1^*$ ,  $\beta_2^*[f, a_1] = b_{2,f}^*(a_1)$  for all  $f \in \mathcal{C}$  and  $a_1 \in A_1$ , and  $\mu^*[\mathcal{C}](f^*) = 1$ . It is easy to check that this assessment induces  $(b_1^*, b_2^*)$  and is a perfect Bayesian equilibrium of  $\Gamma(G)$ .

**[Only if]** Now, suppose that  $(b_1^*, b_2^*)$  can be supported. Then, there exists a perfect Bayesian equilibrium  $(\beta^*, \mu^*)$  that induces  $(b_1^*, b_2^*)$ , i.e.,  $\beta_2^*[\emptyset] = f^*$ ,  $\beta_1^*[\mathcal{C}] = b_1^*$ ,  $\beta_2^*[f^*, a_1] = b_2^*(a_1)$  for all  $a_1 \in A_1$ . The fact that  $(b_1^*, b_2^*)$  is a Nash equilibrium of  $G$  is a direct consequence of sequential rationality of players 1 and 2. We now show that  $f^*$  satisfies conditions 2 and 3 stated in Proposition 1. Suppose, in contradiction to condition 2, that  $f^*(b_2^*(b_1^*)) = \alpha > \delta$  and consider  $f'(a_2) = \delta + (\alpha - \delta)/2$  for all  $a_2$ . This contract is accepted by the third party and  $\beta_2[f', b_1^*] \in \operatorname{argmax}_{a_2} u_2(b_1^*, a_2)$ . Therefore, offering  $f'$  yields player 2 a strictly higher expected payoff than  $f^*$ , a contradiction. Finally, sequential rationality of player 2 immediately implies condition 3.  $\square$

Before we turn to the proof of Theorem 1 we introduce some notation and prove a supplementary lemma. Let the number of elements in  $A_1$  be equal to  $n$  and order its elements so that  $a_1^n \succ_1 a_1^{n-1} \succ_1 \dots \succ_1 a_1^2 \succ_1 a_1^1$ . Let  $e_i$  be the  $i^{\text{th}}$  standard basis row vector for  $\mathbb{R}^n$  and define the row vector  $d_i = e_i - e_{i+1}$ ,  $i = 1, 2, \dots, n-1$ . Let  $D$  be the  $2(n-1) \times n$  matrix whose row  $2i-1$  is  $d_i$  and row  $2i$  is  $-d_i$ ,  $i = 1, \dots, n-1$ . For any  $b_2 \in A_2^{A_1}$  define  $U(b_2)$  as a column vector with  $2(n-1)$  components, where component  $2i-1$  is given by  $u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2(a_1^{i+1}))$  and component  $2i$  is given by  $u_2(a_1^{i+1}, b_2(a_1^{i+1})) - u_2(a_1^{i+1}, b_2(a_1^i))$ ,  $i = 1, 2, \dots, n-1$ .

**Notation 1.** Given two vectors  $x, y \in \mathbb{R}^n$

1.  $x \geq y$  if and only if  $x_i \geq y_i$ , for all  $i = 1, 2, \dots, n$ ;
2.  $x > y$  if and only if  $x_i \geq y_i$ , for all  $i = 1, 2, \dots, n$  and  $x \neq y$ ;
3.  $x \gg y$  if and only if  $x_i > y_i$ , for all  $i = 1, 2, \dots, n$ .

Similarly for  $\leq$ ,  $<$ , and  $\ll$ .

For any  $b_2 \in A_2^{A_1}$  and  $f \in \mathcal{C}$  let  $f(b_2)$  be the column vector with  $n$  components, where  $i^{\text{th}}$  component is given by  $f(b_2(a_1^i))$ ,  $i = 1, 2, \dots, n$ .

It is well-known that if  $b_2$  is increasing, then, under increasing differences, incentive compatibility reduces to local incentive compatibility.<sup>27</sup> We state it as a lemma for future reference.

<sup>27</sup>See, for example, Bolton and Dewatripont (2005), p. 78.

**Lemma 1.** If  $u_2$  has increasing differences and  $b_2 \in A_2^{A_1}$  is increasing in  $(\succsim_1, \succsim_2)$ , then for any  $f \in \mathcal{C}$

$$u_2(a_1^i, b_2(a_1^i)) - f(b_2(a_1^i)) \geq u_2(a_1^i, b_2(a_1^j)) - f(b_2(a_1^j)), \text{ for all } i, j = 1, 2, \dots, n$$

holds if and only if

$$u_2(a_1^i, b_2(a_1^i)) - f(b_2(a_1^i)) \geq u_2(a_1^i, b_2(a_1^{i-1})) - f(b_2(a_1^{i-1})), \text{ for all } i = 2, \dots, n,$$

and

$$u_2(a_1^i, b_2(a_1^i)) - f(b_2(a_1^i)) \geq u_2(a_1^i, b_2(a_1^{i+1})) - f(b_2(a_1^{i+1})), \text{ for all } i = 1, 2, \dots, n-1.$$

*Proof of Theorem 1. [Only if]* Suppose that  $(b_1^*, b_2^*)$  can be supported with incomplete and non-rene-gotiable contracts. Then, there exists a perfect Bayesian equilibrium  $(\beta^*, \mu^*)$  of  $\Gamma(G)$  that induces  $(b_1^*, b_2^*)$ , i.e.,  $\beta_2^*[\emptyset] = f^*$ ,  $\beta_1^*[\mathcal{C}] = b_1^*$ ,  $\beta_2^*[f^*, a_1] = b_2^*(a_1)$  for all  $a_1 \in A_1$ . Given Proposition 1 we only need to prove that  $b_2^*$  is increasing. Fix orders  $(\succsim_1, \succsim_2)$  in which  $u_2$  has strictly increasing differences. Take any  $a_1, a_1' \in A_1$  and assume, without loss of generality, that  $a_1 \succsim_1 a_1'$ . Suppose, for contradiction, that  $b_2^*(a_1') \succ_2 b_2^*(a_1)$ . Sequential rationality of player 2 implies that

$$\begin{aligned} u_2(a_1, b_2^*(a_1)) - f^*(b_2^*(a_1)) &\geq u_2(a_1, b_2^*(a_1')) - f^*(b_2^*(a_1')) \\ u_2(a_1', b_2^*(a_1')) - f^*(b_2^*(a_1')) &\geq u_2(a_1', b_2^*(a_1)) - f^*(b_2^*(a_1)) \end{aligned}$$

and hence

$$u_2(a_1, b_2^*(a_1')) - u_2(a_1, b_2^*(a_1)) \leq u_2(a_1', b_2^*(a_1')) - u_2(a_1', b_2^*(a_1)),$$

contradicting that  $u_2$  has strictly increasing differences. Therefore,  $b_2^*$  must be increasing.

**[If]** Let  $(b_1^*, b_2^*)$  be a Nash equilibrium of  $G$  such that  $b_2^*$  is increasing and  $b_1^* = a_1^k$ , for some  $k = 1, 2, \dots, n$ . Given Proposition 1, all we need to prove is the existence of a contract  $f \in \mathcal{C}$  such that  $f(b_2^*(a_1^k)) = \delta$  and

$$u_2(a_1^i, b_2^*(a_1^i)) - f(b_2^*(a_1^i)) \geq u_2(a_1^i, b_2^*(a_1^j)) - f(b_2^*(a_1^j)), \text{ for all } i, j = 1, 2, \dots, n. \quad (16)$$

By Lemma 1, (16) holds if and only if  $Df(b_2^*) \leq U(b_2^*)$ . Therefore, we need to show that there exists  $f(b_2^*) \in \mathbb{R}^n$  such that  $Ef(b_2^*) \leq V$  where

$$E = \begin{pmatrix} D \\ e_k \\ -e_k \end{pmatrix}, \quad V = \begin{pmatrix} U(b_2^*) \\ \delta \\ -\delta \end{pmatrix}$$

By Gale's theorem for linear inequalities (Mangasarian (1994), p. 33), there exists such an  $f(b_2^*) \in \mathbb{R}^n$  if and only if for any  $y \in \mathbb{R}_+^{2n}$ ,  $E'y = 0$  implies  $y'V \geq 0$ . It is easy to show that  $E'y = 0$  if and only if  $y_1 = y_2, y_3 = y_4, \dots, y_{2n-1} = y_{2n}$ . Let  $U(b_2^*)_i$  denote the  $i^{th}$  row of  $U(b_2^*)$  and note that since  $b_2^*$  is increasing and  $u_2$  has strictly increasing differences,  $U(b_2^*)_{2i-1} + U(b_2^*)_{2i} \geq 0$ , for any  $i = 1, 2, \dots, n-1$ . Therefore,

$$y'V = \sum_{i=1}^{n-1} (U(b_2^*)_{2i-1} + U(b_2^*)_{2i}) y_{2i-1} \geq 0$$

and the proof is completed.  $\square$

*Proof of Proposition 2.* **[If]** Let  $(b_1^*, b_2^*)$  be a Nash equilibrium of  $G$  such that  $b_2^*$  is increasing and renegotiation-proof. This implies that there exists  $f' \in \mathcal{C}$  such that  $(f', b_2^*)$  is incentive compatible and renegotiation-proof. Let  $f^*(b_2^*(a_1)) = f'(b_2^*(a_1)) - f'(b_2^*(b_1^*)) + \delta$  for all  $a_1 \in A_1$  and note that  $f^*(b_2^*(b_1^*)) = \delta$ . Furthermore, using Theorem 2, it can be easily checked that  $(f^*, b_2^*)$  is incentive compatible and renegotiation-proof. For any  $f \neq f^*$  and  $a_1 \in A_1$ , let  $b_{2,f}(a_1) \in \operatorname{argmax}_{a_2} u_2(a_1, a_2) - f(a_2)$  and  $g_{(f,a_1)} \in \operatorname{argmax}_g u_2(a_1, b_{2,g}(a_1)) - g(b_{2,g}(a_1))$  subject to  $g(b_{2,g}(a_1')) \geq f(b_{2,f}(a_1'))$  for all  $a_1'$ .

Consider the following assessment  $(\beta^*, \mu^*)$  of  $\Gamma_R(G)$ :  $\beta_2^*[\emptyset] = f^*$ ;  $\beta_1^*[\mathcal{C}] = b_1^*$ ;  $\beta_2^*[f^*, a_1] = b_{2,f}(a_1)$  for all  $a_1$ ;

$$\beta_2^*[f, a_1] = \begin{cases} g_{(f,a_1)}, & \text{if } u_2(a_1, b_{2,g_{(f,a_1)}}(a_1)) - g_{(f,a_1)}(b_{2,g_{(f,a_1)}}(a_1)) > u_2(a_1, b_{2,f}(a_1)) - f(b_{2,f}(a_1)) \\ b_{2,f}(a_1), & \text{otherwise} \end{cases}$$

for any  $f \neq f^*$  and  $a_1$ ;  $\beta_2^*[f, a_1, g, y] = b_{2,g}(a_1)$  and  $\beta_{2,f}[f, a_1, g, n] = b_2^*(a_1)$  for all  $(a_1, f, g)$ ;

$$\beta_3^*[I_3(f^*, g)] = \begin{cases} y, & \text{if } g(b_{2,g}(a_1)) > f^*(b_2^*(a_1)) \quad \forall a_1 \\ n, & \text{otherwise} \end{cases}$$

and

$$\beta_3^*[I_3(f, g)] = \begin{cases} y, & \text{if } g(b_{2,g}(a_1)) \geq f(b_{2,f}(a_1)) \quad \forall a_1 \\ n, & \text{otherwise} \end{cases}$$

for any  $g$  and  $f \neq f^*$ ;  $\mu^*[\mathcal{C}](f^*) = 1$ ; For any  $g$ ,  $\mu^*[I_3(f^*, g)](b_1^*) = 1$  if  $g(b_{2,g}(a_1)) > f^*(b_2^*(a_1))$  for all  $a_1$  and  $\mu^*[I_3(f^*, g)](a_1') = 1$  if there exists  $a_1'$  such that  $f^*(b_2^*(a_1')) \geq g(b_{2,g}(a_1'))$ ; For any  $f \neq f^*$  and  $g$ ,  $\mu^*[I_3(f, g)](b_1^*) = 1$  if  $g(b_{2,g}(a_1)) \geq f(b_{2,f}(a_1))$  for all  $a_1$  and  $\mu^*[I_3(f, g)](a_1') = 1$  if there exists  $a_1'$  such that  $f(b_{2,f}(a_1')) > g(b_{2,g}(a_1'))$ . It is easy to check that this assessment induces  $(b_1^*, b_2^*)$  and is a renegotiation-proof perfect Bayesian equilibrium.

**[Only if]** Suppose that  $\Gamma_R(G)$  has a renegotiation-proof perfect Bayesian equilibrium  $(\beta^*, \mu^*)$  that induces  $(b_1^*, b_2^*)$ . Letting  $\beta_2^*[\emptyset] = f^*$ , we have  $\beta_1^*[\mathcal{C}] = b_1^*$ ,  $\beta_2[f^*, a_1] = b_2^*(a_1)$  for all  $a_1$ , and  $\mu^*[\mathcal{C}](f^*) = 1$ . Sequential rationality of player 1 implies that

$$b_1^* \in \operatorname{argmax}_{a_1} u_1(a_1, b_2^*(a_1)) \quad (17)$$

whereas that of player 2 implies  $u_2(a_1, b_2^*(a_1)) - f^*(b_2^*(a_1)) \geq u_2(a_1, b_2^*(a_1')) - f^*(b_2^*(a_1'))$  for all  $a_1, a_1' \in A_1$ , which, together with increasing differences, implies that  $b_2^*$  is increasing.

We also claim that

$$b_2^*(b_1^*) \in \operatorname{argmax}_{a_2} u_2(b_1^*, a_2) \quad (18)$$

Suppose, for contradiction, that this is not the case and let  $\hat{a}_2 \in \operatorname{argmax}_{a_2} u_2(b_1^*, a_2)$  and define  $\varepsilon = u_2(b_1^*, \hat{a}_2) - u_2(b_1^*, b_2^*(b_1^*)) > 0$ . Define  $f'(a_2) = f^*(b_2^*(b_1^*)) + \varepsilon/2$  and note that the third party accepts  $f'$ . Assume first that  $\beta_2^*[f', b_1^*] \in A_2$ , i.e.,  $f'$  is not renegotiated after  $b_1^*$  and note that sequential rationality of player 2 implies that  $\beta_2^*[f', b_1^*] \in \operatorname{argmax}_{a_2} u_2(b_1^*, a_2)$ . Therefore, player 2's payoff under

$f'$  is

$$u_2(b_1^*, \hat{a}_2) - f^*(b_2^*(b_1^*)) - \varepsilon/2 > u_2(b_1^*, b_2^*(b_1^*)) - f^*(b_2^*(b_1^*))$$

contradicting that  $(\beta^*, \mu^*)$  is a PBE. Now assume that  $f'$  is renegotiated after  $b_1^*$ . This implies that there exists an incentive compatible  $(g, b_2)$  such that  $\beta_2^*[f', b_1^*] = g$ ,  $\beta_3^*[f', b_1^*, g] = y$ ,  $\beta_2^*[f', b_1^*, g, y] = b_2$ .<sup>28</sup> Therefore, letting  $b_{2,f'}(b_1^*) \in \operatorname{argmax}_{a_2} u_2(b_1^*, a_2) - f'(a_2)$ ,

$$\begin{aligned} u_2(b_1^*, b_2(b_1^*)) - g(b_2(b_1^*)) &\geq u_2(b_1^*, b_{2,f'}(b_1^*)) - f'(b_{2,f'}(b_1^*)) \\ g(b_2(b_1^*)) &\geq f'(b_{2,f'}(b_1^*)) \end{aligned}$$

which implies that  $b_2(b_1^*) \in \operatorname{argmax}_{a_2} u_2(b_1^*, a_2)$ . Player 2's payoff under  $f'$  is

$$\begin{aligned} u_2(b_1^*, b_2(b_1^*)) - g(b_2(b_1^*)) &\geq u_2(b_1^*, b_{2,f'}(b_1^*)) - f'(b_{2,f'}(b_1^*)) \\ &= u_2(b_1^*, \hat{a}_2) - f^*(b_2^*(b_1^*)) - \varepsilon/2 \\ &> u_2(b_1^*, b_2^*(b_1^*)) - f^*(b_2^*(b_1^*)) \end{aligned}$$

contradicting that  $(\beta^*, \mu^*)$  is a PBE.

Therefore, by (17) and (18),  $(b_1^*, b_2^*)$  is a Nash equilibrium of  $G$  and  $b_2^*$  is increasing. Finally, suppose that  $b_2^*$  is not renegotiation-proof. This implies that for any contract  $f$  such that  $(f, b_2^*)$  is incentive compatible, there exists an  $a_1'$  and an incentive compatible  $(g, b_2)$  such that  $u_2(a_1', b_2(a_1')) - g(b_2(a_1')) > u_2(a_1', b_2^*(a_1')) - f(b_2^*(a_1'))$  and  $g(b_2(a_1')) > f(b_2^*(a_1'))$  for all  $a_1$ . This implies that, in any perfect Bayesian equilibrium, after history  $(f, a_1')$  player 2 strictly prefers to renegotiate and offer  $g$  and the third party accepts it. In other words, there exists no renegotiation-proof perfect Bayesian equilibrium which induces  $(b_1^*, b_2^*)$ , completing the proof.  $\square$

*Proof of Theorem 2.* By definition  $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1}$  is not renegotiation-proof if and only if there exist  $i = 1, 2, \dots, n$  and incentive compatible  $(g, b_2) \in \mathcal{C} \times A_2^{A_1}$  such that  $u_2(a_1^i, b_2(a_1^i)) - g(b_2(a_1^i)) > u_2(a_1^i, b_2^*(a_1^i)) - f(b_2^*(a_1^i))$  and  $g(b_2(a_1^i)) > f(b_2^*(a_1^i))$  for all  $j = 1, 2, \dots, n$ . For any  $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1}$ , let  $f(b_2^*) \in \mathbb{R}^n$  be a vector whose row  $j = 1, 2, \dots, n$  is given by  $f(b_2^*(a_1^j))$ . Note that incentive compatibility of  $(g, b_2) \in \mathcal{C} \times A_2^{A_1}$  is equivalent to  $Dg(b_2) \leq U(b_2)$ . Therefore,  $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1}$  is not renegotiation-proof if and only if there exist  $i = 1, 2, \dots, n$  and  $(g(b_2), b_2) \in \mathbb{R}^n \times A_2^{A_1}$  such that  $Dg(b_2) \leq U(b_2)$ ,  $u_2(a_1^i, b_2(a_1^i)) - g(b_2(a_1^i)) > u_2(a_1^i, b_2^*(a_1^i)) - f(b_2^*(a_1^i))$ , and  $g(b_2) \gg f(b_2^*)$ . Also note that  $g(b_2) \gg f(b_2^*)$  if and only if there exists an  $\varepsilon \gg 0$  such that  $g(b_2) = f(b_2^*) + \varepsilon$ . Therefore, we have the following

**Lemma 2.**  $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1}$  is not renegotiation-proof if and only if there exist  $i = 1, 2, \dots, n$ ,  $b_2 \in A_2^{A_1}$ , and  $\varepsilon \in \mathbb{R}^n$  such that  $D(f(b_2^*) + \varepsilon) \leq U(b_2)$ ,  $\varepsilon_i < u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i))$ , and  $\varepsilon \gg 0$ .

We first state a theorem of the alternative, which we will use in the sequel.

**Lemma 3** (Motzkin's Theorem). *Let  $A$  and  $C$  be given matrices, with  $A$  being non-vacuous. Then either*

1.  $Ax \gg 0$  and  $Cx \geq 0$  has a solution  $x$

or

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<sup>28</sup>We do not consider the case in which  $\beta_3^*[f', b_1^*, g] = n$  since this is equivalent to the case  $\beta_2^*[f', b_1^*] \in A_2$ .

2.  $A'y_1 + C'y_2 = 0$ ,  $y_1 > 0$ ,  $y_2 \geq 0$  has a solution  $y_1, y_2$

but not both.

*Proof of Lemma 3.* See Mangasarian (1994), p. 28.  $\square$

For any  $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1}$ ,  $b_2 \in A_2^{A_1}$ , and  $i = 1, 2, \dots, n$ , define  $V = U(b_2) - Df(b_2^*)$ ,  $C = \begin{pmatrix} V & -D \end{pmatrix}$ , and

$$A = \begin{pmatrix} I_{n+1} \\ l_i \end{pmatrix}$$

where  $l_i = (u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)))e_1 - e_{i+1}$ . Note that  $C$  and  $A$  depend on and are uniquely defined by  $(f, b_2^*)$  and  $(i, b_2)$  but we suppress this dependency for notational convenience. The following lemma uses Motzkin's Theorem to express renegotiation-proofness as an alternative.

**Lemma 4.**  $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1}$  is renegotiation-proof if and only if for any  $i = 1, 2, \dots, n$  and  $b_2 \in A_2^{A_1}$  there exist  $y \in \mathbb{R}^{n+2}$  and  $z \in \mathbb{R}^{2(n-1)}$  such that  $A'y + C'z = 0$ ,  $y > 0$ ,  $z \geq 0$ .

*Proof of Lemma 4.* By Lemma 2,  $(f, b_2^*)$  is not renegotiation-proof if and only if there exist  $i = 1, 2, \dots, n$ ,  $b_2 \in A_2^{A_1}$ , and  $\varepsilon \in \mathbb{R}^n$  such that  $D(f(b_2^*) + \varepsilon) \leq U(b_2)$ ,  $\varepsilon_i < u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i))$ , and  $\varepsilon \gg 0$ . This is true if and only if for some  $i$  and  $b_2$  there exists an  $x \in \mathbb{R}^{n+1}$  such that  $Ax \gg 0$  and  $Cx \geq 0$ . To see this let  $\xi > 0$  and define

$$x = \begin{pmatrix} \xi \\ \xi \varepsilon \end{pmatrix}$$

Then  $D(f(b_2^*) + \varepsilon) \leq U(b_2)$  if and only if  $Cx \geq 0$ . Also,  $\varepsilon \gg 0$  and  $\varepsilon_i < u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i))$  if and only if  $Ax \gg 0$ . The lemma then follows from Motzkin's Theorem.  $\square$

For any  $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1}$ ,  $b_2 \in A_2^{A_1}$ , and  $i = 1, 2, \dots, n$ , let  $U(b_2)_j$  denote the  $j$ -th row of vector  $U(b_2)$  and define  $\alpha_1 = 1$ ,  $\alpha_{i+1} = u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i))$ , and

$$\alpha_{k+1} = \sum_{j=k}^{i-1} U(b_2)_{2j-1} + \alpha_{i+1} - f(b_2^*(a_1^k)) + f(b_2^*(a_1^i)), \quad \text{for } k = 1, 2, \dots, i-1,$$

$$\alpha_{l+1} = \sum_{j=i+1}^l U(b_2)_{2(j-1)} + \alpha_{i+1} - f(b_2^*(a_1^l)) + f(b_2^*(a_1^i)), \quad \text{for } l = i+1, i+2, \dots, n,$$

$$\beta_j = U(b_2)_{2j} + U(b_2)_{2j-1}, \quad \text{for } j = 1, 2, \dots, n-1.$$

Again, note that  $\alpha_j$  and  $\beta_j$  depend on and are uniquely defined by  $(f, b_2^*)$  and  $(i, b_2)$  but we suppress this dependency. We have the following lemma.

**Lemma 5.** For any  $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1}$ ,  $b_2 \in A_2^{A_1}$ , and  $i = 1, 2, \dots, n$ , there exist  $y \in \mathbb{R}^{n+2}$  and  $z \in \mathbb{R}^{2(n-1)}$  such that  $A'y + C'z = 0$ ,  $y > 0$ , and  $z \geq 0$  if and only if there exist  $\hat{y} \in \mathbb{R}^{n+1}$  and  $\hat{z} \in \mathbb{R}^{(n-1)}$  such that  $\hat{y} > 0$ ,  $\hat{z} \geq 0$ , and

$$\sum_{j=1}^{n+1} \alpha_j \hat{y}_j + \sum_{j=1}^{n-1} \beta_j \hat{z}_j = 0 \quad (19)$$

*Proof of Lemma 5.* Fix  $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1}$ ,  $b_2 \in A_2^{A_1}$ , and  $i = 1, 2, \dots, n$ . First note that for any  $y$  and  $z$ ,  $A'y + C'z = 0$  if and only if

$$y_1 + (u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)))y_{n+2} + V'z = 0 \quad (20)$$

$$D'z = [A'y]_{-1} \quad (21)$$

where  $[A'y]_{-1}$  is the  $n$ -dimensional vector obtained from  $A'y$  by eliminating the first row. Recursively adding row 1 to row 2, row 2 to row 3, and so on, we can reduce  $(D' \quad [A'y]_{-1})$  to a row echelon form and show that (21) holds if and only if

$$z_{2j-1} = z_{2j} + \sum_{k=1}^j y_{k+1}, \quad j = 1, 2, \dots, i-1 \quad (22)$$

$$z_{2j} = z_{2j-1} + \sum_{k=j+1}^n y_{k+1}, \quad j = i, i+1, \dots, n-1 \quad (23)$$

$$y_{n+2} = \sum_{k=1}^n y_{k+1} \quad (24)$$

Substituting (21)-(24) into (20) we get

$$\begin{aligned} y_1 + \alpha_{i+1} \sum_{k=1}^n y_{k+1} + \sum_{j=1}^{i-1} U(b_2)_{2j-1} \sum_{k=1}^j y_{k+1} + \sum_{j=i}^{n-1} U(b_2)_{2j} \sum_{k=j+1}^n y_{k+1} + \sum_{j=1}^{i-1} (U(b_2)_{2j-1} + U(b_2)_{2j}) z_{2j} \\ + \sum_{j=i}^{n-1} (U(b_2)_{2j-1} + U(b_2)_{2j}) z_{2j-1} - \sum_{k=1}^n (f(b_2^*(a_1^k)) - f(b_2^*(a_1^i))) y_{k+1} = 0 \end{aligned} \quad (25)$$

Therefore,  $A'y + C'z = 0$  if and only if equations (22) through (25) hold. Now suppose that there exist  $y \in \mathbb{R}^{n+2}$  and  $z \in \mathbb{R}^{2(n-1)}$  such that  $y > 0$ ,  $z \geq 0$ , and (22) through (25) hold. Define  $\hat{y}_j = y_j$ , for  $j = 1, \dots, n+1$  and

$$\hat{z}_j = \begin{cases} z_{2j}, & j = 1, \dots, i-1 \\ z_{2j-1}, & j = i, \dots, n-1 \end{cases}$$

It is easy to verify that  $\hat{y} > 0$ ,  $\hat{z} \geq 0$ , and  $\sum_{j=1}^{n+1} \alpha_j \hat{y}_j + \sum_{j=1}^{n-1} \beta_j \hat{z}_j = 0$ .

Conversely, suppose that there exist  $\hat{y} \in \mathbb{R}^{n+1}$  and  $\hat{z} \in \mathbb{R}^{(n-1)}$  such that  $\hat{y} > 0$ ,  $\hat{z} \geq 0$ , and (19) holds. Define  $y_j = \hat{y}_j$  for  $j = 1, \dots, n+1$  and  $y_{n+2} = \sum_{i=1}^{n+1} \hat{y}_j$ . For any  $j = 1, \dots, i-1$ , let  $z_{2j-1} = \hat{z}_j + \sum_{k=1}^j y_{k+1}$  and  $z_{2j} = \hat{z}_j$ , and for any  $j = i, \dots, n-1$ , let  $z_{2j-1} = \hat{z}_j$  and  $z_{2j} = \hat{z}_j + \sum_{k=j+1}^n y_{k+1}$ . It is straightforward to show that  $y > 0$ ,  $z \geq 0$ , and (22) through (25) hold. This completes the proof of Lemma 5.  $\square$

Lemmas 4 and 5 imply that  $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1}$  is renegotiation-proof if and only if for any  $i \in \{1, 2, \dots, n\}$  and  $b_2 \in A_2^{A_1}$ , there exist  $\hat{y} \in \mathbb{R}^{n+1}$  and  $\hat{z} \in \mathbb{R}^{(n-1)}$  such that  $\hat{y} > 0$ ,  $\hat{z} \geq 0$ , and equation (19) holds. We can now complete the proof of Theorem 2.

**[Only if]** Suppose, for contradiction, that there exist  $i = 1, 2, \dots, n$  and an increasing  $b_2 \in A_2^{A_1}$  such that  $u_2(a_1^i, b_2(a_1^i)) > u_2(a_1^i, b_2^*(a_1^i))$ , but there is no  $k = 1, 2, \dots, i-1$  such that (5) holds and no  $l = i+1, \dots, n$  such that (6) holds. This implies that  $\alpha_j > 0$  for all  $j = 1, \dots, n+1$ . Since  $u_2$  has increasing differences,  $\beta_j \geq 0$  for all  $j = 1, \dots, n-1$ . Therefore,  $\hat{y} > 0$  and  $\hat{z} \geq 0$  imply that  $\sum_{j=1}^{n+1} \alpha_j \hat{y}_j + \sum_{j=1}^{n-1} \beta_j \hat{z}_j > 0$ , which, by Lemma 5, contradicts that  $(f, b_2^*)$  is renegotiation-proof.

**[If]** Fix arbitrary  $i = 1, 2, \dots, n$  and increasing  $b_2 \in A_2^{A_1}$  such that  $u_2(a_1^i, b_2(a_1^i)) > u_2(a_1^i, b_2^*(a_1^i))$ . Sup-

pose first that there exists a  $k \in \{1, \dots, i-1\}$  such that (5) holds. This implies that  $\alpha_{i+1} > 0$  and  $\alpha_{k+1} \leq 0$ . Let  $\hat{y}_{k+1} = 1$ ,  $\hat{y}_{i+1} = \frac{-\alpha_{k+1}}{\alpha_{i+1}} \geq 0$ , and all the other  $\hat{y}_j = 0$  and  $\hat{z}_j = 0$ . This implies that equation (19) holds and, by Lemma 5, that  $(f, b_2^*)$  is renegotiation-proof. Suppose now that there exists an  $l \in \{i+1, \dots, n\}$  such that (6) holds. Then,  $\alpha_{i+1} > 0$  and  $\alpha_{l+1} \leq 0$ . Let  $\hat{y}_{l+1} = 1$ ,  $\hat{y}_{i+1} = \frac{-\alpha_{l+1}}{\alpha_{i+1}} \geq 0$  and all the other  $\hat{y}_j = 0$  and  $\hat{z}_j = 0$ . This, again, implies that (19) holds and that  $(f, b_2^*)$  is renegotiation-proof.  $\square$

*Proof of Proposition 3.* Suppose that  $b_2^*$  is renegotiation-proof and fix an  $i = 1, \dots, n$  and a  $b_2^i \in \mathfrak{B}(i, b_2^*)$ . For any  $j = 1, \dots, n$ , let  $c_j = e_i - e_j$ , where  $e_j$  is the  $j^{\text{th}}$  standard basis row vector for  $\mathbb{R}^n$ , and define

$$E_j = \begin{pmatrix} D \\ c_j \end{pmatrix}$$

Also let

$$w_k = u_2(a_1^i, b_2^i(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \sum_{j=k}^{i-1} U(b_2^i)_{2j-1}$$

$$w_l = u_2(a_1^i, b_2^i(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \sum_{j=i+1}^l U(b_2^i)_{2(j-1)}$$

for any  $k \in \{1, \dots, i-1\}$  and  $l \in \{i+1, \dots, n\}$  and define

$$V_j = \begin{pmatrix} U(b_2^*) \\ -w_j \end{pmatrix}$$

Incentive compatibility of  $(f, b_2^*)$  implies that  $Df(b_2^*) \leq 0$ . Renegotiation proofness, by Theorem 2, implies that  $c_k f(b_2^*) \leq -w_k$  for some  $k \in \{1, \dots, i-1\}$  or  $c_l f(b_2^*) \leq -w_l$  for some  $l \in \{i+1, \dots, n\}$ . Suppose first that there exists a  $k \in \{1, \dots, i-1\}$  such that  $c_k f(b_2^*) \leq -w_k$ . Then we must have  $E_k f(b_2^*) \leq V_k$ . By Gale's theorem of linear inequalities, this implies that  $x \geq 0$  and  $E'x = 0$  implies  $x'V_k \geq 0$ . Denote the first  $2(n-1)$  elements of  $x$  by  $y$  and the last element by  $z$ . It is easy to show that  $E'x = 0$  implies that  $y_{2j-1} = y_{2j} + z$  for  $j \in \{k, k+1, \dots, i-1\}$  and  $y_{2j-1} = y_{2j}$  for  $j \notin \{k, k+1, \dots, i-1\}$ . Therefore,

$$\begin{aligned} x'V_k &= \sum_{j=1}^{n-1} U(b_2^*)_{2j} y_{2j} + \sum_{j=1}^{n-1} U(b_2^*)_{2j-1} y_{2j-1} - \sum_{b_2^i} z w_k \\ &= \sum_{j=1}^{n-1} (U(b_2^*)_{2j} + U(b_2^*)_{2j-1}) y_{2j} + \sum_k^{i-1} z (-w_k + \sum_{j=k}^{i-1} U(b_2^*)_{2j-1}) \\ &\geq 0 \end{aligned}$$

Increasing differences imply that  $-w_k + \sum_{j=k}^{i-1} U(b_2^*)_{2j-1} \geq 0$  and hence  $k$  is a blocking action.

Similarly, we can show that, if there exists an  $l \in \{i+1, \dots, n\}$  such that  $c_l f(b_2^*) \leq -w_l$ , then  $l$  is a blocking action, and this completes the proof.  $\square$

*Proof of Proposition 4.* We will show that there exists an  $f \in \mathcal{C}$  such that  $(f, b_2^*)$  is incentive compatible and renegotiation-proof. For any  $i = 1, \dots, n$  and  $b_2^i \in \mathfrak{B}(i, b_2^*)$  pick a blocking action  $m(b_2^i)$  that satisfies the conditions of the proposition. Let  $c_{b_2^i} = e_i - e_{m(b_2^i)}$  for each  $i$  and  $b_2^i \in \mathfrak{B}(i, b_2^*)$ , and let

$\sum_i |\mathfrak{B}(i, b_2^*)| \times n$  matrix  $C$  have row  $c_{b_2^i}$  corresponding to each  $b_2^i$ . Let  $E$  be given by

$$E = \begin{pmatrix} D \\ C \end{pmatrix}$$

Also let

$$w_{b_2^i} = u_2(a_1^i, b_2^i(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \mathbf{1}_{\{m(b_2^i) \leq i-1\}} \sum_{j=m(b_2^i)}^{i-1} U(b_2^i)_{2j-1} + \mathbf{1}_{\{i \leq m(b_2^i)-1\}} \sum_{j=i+1}^{m(b_2^i)} U(b_2^i)_{2(j-1)}$$

and  $\sum_i |\mathfrak{B}(i, b_2^*)| \times 1$  vector  $W$  have row  $w_{b_2^i}$  corresponding to each  $b_2^i$ . Define

$$V = \begin{pmatrix} U(b_2^*) \\ -W \end{pmatrix}$$

Observe that if  $Ef(b_2^*) \leq V$ , then  $Df(b_2^*) \leq U(b_2^*)$ , and hence  $(f, b_2^*)$  is incentive compatible. Furthermore,  $Ef(b_2^*) \leq V$  implies  $W \leq -Cf(b_2^*)$ , and, by Theorem 2, that  $(f, b_2^*)$  is renegotiation-proof. Therefore, if we can show that there exists  $f(b_2^*) \in \mathbb{R}^n$  such that  $Ef(b_2^*) \leq V$ , the proof would be completed. By Gale's theorem of linear inequalities this is equivalent to showing  $x \geq 0$  and  $E'x = 0$  implies  $x'V \geq 0$ . Decompose  $x$  into two vectors so that the first  $2(n-1)$  elements constitute  $y$  and the remaining  $\sum_i |\mathfrak{B}(i, b_2^*)|$  components constitute  $z$ . Notice that for any  $i = 1, \dots, n$  and  $b_2^i \in \mathfrak{B}(i, b_2^*)$  there is a corresponding element of  $z$ , which we will denote  $z_{b_2^i}$ .

Recursively adding row 1 to row 2, row 2 to row 3, and so on, we can reduce  $E'$  to a row echelon form and show that  $E'x = 0$  if and only if

$$y_{2j-1} = y_{2j} + \sum_{b_2^i} z_{b_2^i} [\mathbf{1}_{\{m(b_2^i) \leq j \leq i-1\}} - \mathbf{1}_{\{i \leq j \leq m(b_2^i)-1\}}] \quad (26)$$

for  $j = 1, \dots, n-1$ .

Let  $J_- = \{j \in \{1, \dots, n-1\} : \exists b_2^i \text{ such that } i \leq j \leq m(b_2^i)-1\}$  and  $J_+ = \{j \in \{1, \dots, n-1\} : \exists b_2^i \text{ such that } m(b_2^i) \leq j \leq i-1\}$  and note that  $J_- \cap J_+ = \emptyset$ . To see this, suppose, for contradiction, that there exists a  $j \in J_- \cap J_+$ . Therefore, there exists a  $b_2^i$  such that  $i \leq j \leq m(b_2^i)-1$  and  $b_2^{i'}$  such that  $m(b_2^{i'}) \leq j \leq i'-1$ . This implies that  $i < i'$ ,  $m(b_2^i) > i$ ,  $m(b_2^{i'}) < i'$ , but  $m(b_2^i) > m(b_2^{i'})$ , contradicting the conditions of the proposition. We can therefore write (26) as

$$y_{2j} = y_{2j-1} + \sum_{b_2^i} z_{b_2^i} \mathbf{1}_{\{i \leq j \leq m(b_2^i)-1\}} \quad (27)$$

for  $j \in J_-$  and

$$y_{2j-1} = y_{2j} + \sum_{b_2^i} z_{b_2^i} \mathbf{1}_{\{m(b_2^i) \leq j \leq i-1\}} \quad (28)$$

for  $j \in J_+$ .

Finally note that

$$x'V = \sum_{j=1}^{n-1} U(b_2^*)_{2j} y_{2j} + \sum_{j=1}^{n-1} U(b_2^*)_{2j-1} y_{2j-1} - \sum_{b_2^i} z_{b_2^i} w_{b_2^i}$$

Substituting from (27) and (28) we obtain

$$x'V = \sum_{j \in J_-} [U(b_2^*)_{2j} + U(b_2^*)_{2j-1}] y_{2j-1} + \sum_{j \in J_+} [U(b_2^*)_{2j} + U(b_2^*)_{2j-1}] y_{2j} \\ + \sum_{b_2^i} z_{b_2^i} \left[ -w_{b_2^i} + \mathbf{1}_{\{m(b_2^i) \leq i-1\}} \sum_{j=m(b_2^i)}^{i-1} U(b_2^*)_{2j-1} + \mathbf{1}_{\{i \leq m(b_2^i)-1\}} \sum_{j=i}^{m(b_2^i)-1} U(b_2^*)_{2j} \right]$$

Increasing differences, the definition of  $m(b_2^i)$ , and  $y, z \geq 0$  imply that  $x'V \geq 0$ , and the proof is completed.  $\square$

*Proof of Theorem 3.* Before we proceed to the proof of the theorem, we first introduce some definitions and prove an intermediate lemma.

**Definition 13.** For any  $b_2 \in A_2^{A_1}$  we say that  $i \in \{1, 2, \dots, n\}$  has *right (left) deviation* at  $b_2$  if there exists an  $a_2 \in A_2$  such that  $a_2 \succsim_2 (\succ_2) b_2(a_1^i)$  and  $u_2(a_1^i, a_2) > u_2(a_1^i, b_2(a_1^i))$ . Otherwise, we say that  $i$  has *no right (left) deviation* at  $b_2$ .

Let  $BR_j(a_{-j}) = \operatorname{argmax}_{a_j} u_j(a_j, a_{-j})$ , for  $j = 1, 2$ . For any  $b_2 \in A_2^{A_1}$  and  $i \in \{1, \dots, n\}$  that has right deviation at  $b_2$  define

$$R(i) = \{k > i : b_2(a_1^k) \in BR_2(a_1^k) \text{ and } i < j < k \text{ implies that } j \text{ has no left deviation at } b_2\}.$$

Similarly, for  $i \in \{1, \dots, n\}$  that has left deviation at  $b_2$  let

$$L(i) = \{k < i : b_2(a_1^k) \in BR_2(a_1^k) \text{ and } k < j < i \text{ implies that } j \text{ has no right deviation at } b_2\},$$

**Lemma 6.**  $b_2^*$  is renegotiation-proof if for any  $i_1$  ( $i_2$ ) that has right (left) deviation at  $b_2^*$ ,  $R(i_1) \neq \emptyset$  ( $L(i_2) \neq \emptyset$ ), and  $i_1 < i_2$  implies  $R(i_1) \cap L(i_2) \neq \emptyset$ .

*Proof of Lemma 6.* Fix an  $i \in \{1, \dots, n\}$  and  $b_2^i \in \mathfrak{B}(i, b_2^*)$ . Assume first that  $b_2^i(a_1^i) \succsim_2 b_2^*(a_1^i)$  and note that  $R(i) \neq \emptyset$  by assumption. Let  $J = \{i+1 \leq j \leq \min R(i) - 1 : b_2^*(a_1^j) \succ_2 b_2^i(a_1^j)\}$ . If  $J = \emptyset$ , let  $m(b_2^i) = \min R(i)$  and if  $J \neq \emptyset$ , let  $m(b_2^i) = \min J$  and note that we have

$$\sum_{j=i+1}^{m(b_2^i)} \left( u_2(a_1^j, b_2^i(a_1^{j-1})) - u_2(a_1^j, b_2^*(a_1^{j-1})) - [u_2(a_1^{j-1}, b_2^i(a_1^{j-1})) - u_2(a_1^{j-1}, b_2^*(a_1^{j-1}))] \right) \\ + u_2(a_1^{m(b_2^i)}, b_2^*(a_1^{m(b_2^i)})) - u_2(a_1^{m(b_2^i)}, b_2^i(a_1^{m(b_2^i)})) \geq 0 \quad (29)$$

which implies that  $m(b_2^i)$  is a blocking action.

Assume now that  $b_2^*(a_1^i) \succsim_2 b_2^i(a_1^i)$  and note that  $L(i) \neq \emptyset$ . Let  $J = \{\max L(i) + 1 \leq j \leq i-1 : b_2^i(a_1^j) \succ_2 b_2^*(a_1^j)\}$ . If  $J = \emptyset$ , let  $m(b_2^i) = \max L(i)$  and if  $J \neq \emptyset$ , let  $m(b_2^i) = \max J$  and note that

$$\sum_{j=m(b_2^i)}^{i-1} \left( u_2(a_1^{j+1}, b_2^*(a_1^{j+1})) - u_2(a_1^{j+1}, b_2^i(a_1^{j+1})) - [u_2(a_1^j, b_2^*(a_1^{j+1})) - u_2(a_1^j, b_2^i(a_1^{j+1}))] \right) \\ + u_2(a_1^{m(b_2^i)}, b_2^*(a_1^{m(b_2^i)})) - u_2(a_1^{m(b_2^i)}, b_2^i(a_1^{m(b_2^i)})) \geq 0 \quad (30)$$

which, again, implies that  $m(b_2^i)$  is a blocking action.

Finally, suppose that there exist  $i_1 < i_2$  such that  $m(b_2^{i_1}) > i_1$  and  $m(b_2^{i_2}) < i_2$ . This implies that  $i_1$  has right deviation and  $i_2$  has left deviation at  $b_2^*$ , and hence  $R(i_1) \cap L(i_2) \neq \emptyset$ . But this implies that  $m(b_2^{i_1}) \leq m(b_2^{i_2})$  and the proof is completed by applying Proposition 4.  $\square$

We can now proceed to the proof of Theorem 3.

**[If]** Fix an  $(a_1^*, a_2^*)$ , where  $a_1^* \succsim_1 \bar{a}_1^{NE}$ , and a selection  $(br_1, br_2)$  such that  $a_2^* = br_2(a_1^*)$  and (9) is satisfied. Define

$$b_2(a_1) = \begin{cases} \underline{a}_2, & a_1 <_1 a_1^* \\ br_2(a_1^*), & a_1^* \succsim_1 a_1 <_1 \bar{a}_1 \\ br_2(\bar{a}_1), & a_1 = \bar{a}_1 \end{cases}$$

First, note that  $b_2$  is increasing and  $b_2(a_1^*) = a_2^*$ . Second, since  $u_2$  is single peaked,  $b_2$  satisfies the conditions of Lemma 6 and hence is renegotiation proof. Therefore, by Proposition 2, all we need to do is to show that  $(a_1^*, b_2)$  is a Nash equilibrium of the original game  $G$ . By definition  $b_2(a_1^*) \in BR_2(a_1^*)$ . Condition (9) implies that  $u_1(a_1^*, b_2(a_1^*)) \geq u_1(a_1, b_2(a_1))$  for all  $a_1 <_1 a_1^*$  and  $u_1(a_1^*, b_2(a_1^*)) \geq u_1(\bar{a}_1, b_2(\bar{a}_1))$ . Therefore, take any  $a_1$  such that  $a_1^* <_1 a_1 <_1 \bar{a}_1$ . Since  $NE(S(G))$  is stable,  $a_1 >_1 a_1^* \succsim_1 br_1(br_2(a_1^*))$ , which, together with single-peakedness, implies that

$$u_1(a_1^*, b_2(a_1^*)) = u_1(a_1^*, br_2(a_1^*)) \geq u_1(a_1, br_2(a_1^*)) = u_1(a_1, b_2(a_1)).$$

Therefore,  $a_1^* \in \operatorname{argmax}_{a_1} u_1(a_1, b_2(a_1))$  and hence  $(a_1^*, b_2)$  is a Nash equilibrium of  $G$ .

**[Only if]** Suppose that  $(a_1^*, a_2^*) \in A_1 \times A_2$  can be supported with incomplete and renegotiable contracts. This, by Theorem 1, implies that there exists an increasing  $b_2 \in A_2^{A_1}$  such that  $(a_1^*, b_2)$  is a Nash equilibrium of  $G$  and  $b_2(a_1^*) = a_2^*$ . This, in turn, implies that there exists a  $br_2 \in BR_2$  such that  $a_2^* = br_2(a_1^*)$ .

Suppose, for contradiction, that  $a_1^* <_1 \underline{a}_1^{NE}$ . Stability of  $NE(S(G))$  implies that  $a_1^* \succsim_1 br_1(a_2^*)$ , for any  $br_1$ . Fix a  $br_1$  and let  $a_1' = br_1(a_2^*)$ . Note that  $a_1' \succsim_1 a_1^*$  and  $u_1(a_1^*, a_2^*) < u_1(a_1', a_2^*)$ , for otherwise the game  $S(G)$  would have a Nash equilibrium in which player 1's action is smaller than  $\underline{a}_1^{NE}$ . Therefore,

$$u_1(a_1^*, b_2(a_1^*)) = u_1(a_1^*, a_2^*) < u_1(a_1', a_2^*) = u_1(a_1', b_2(a_1^*)) \leq u_1(a_1', b_2(a_1')),$$

where the last inequality follows from positive externality. This contradicts that  $(a_1^*, b_2)$  is a Nash equilibrium of  $G$ .

Choose  $br_2 \in BR_2$  such that  $a_2^* = br_2(a_1^*) = b_2(a_1^*)$  and take any  $br_1 \in BR_1$ . Suppose, for contradiction, that (9) is not satisfied for this selection of  $(br_1, br_2)$ . If  $u_1(br_1(\underline{a}_2), \underline{a}_2) > u_1(a_1^*, br_2(a_1^*))$ , then there exists  $a_1'$  such that  $u_1(a_1', \underline{a}_2) > u_1(a_1^*, br_2(a_1^*))$ . This implies that

$$u_1(a_1', b_2(a_1')) \geq u_1(a_1', \underline{a}_2) > u_1(a_1^*, br_2(a_1^*)) = u_1(a_1^*, b_2(a_1^*)),$$

where the first inequality follows from positive externality and that  $b_2$  is increasing. This contradicts that  $(a_1^*, b_2)$  is a Nash equilibrium.

To prove that  $u_1(a_1^*, br_2(a_1^*)) \geq u_1(\bar{a}_1, br_2(\bar{a}_1))$ , we first prove the following lemma.

**Lemma 7.** *If  $b_2 \in A_2^{A_1}$  is renegotiation-proof, then  $\bar{a}_1$  does not have right deviation.*<sup>29</sup>

<sup>29</sup>See Definition 13.

*Proof of Lemma 7.* Let  $a_1^n = \bar{a}_1$  and suppose, for contradiction, that  $a_1^n$  has right deviation, i.e., there exists  $a_2' >_2 b_2(a_1^n)$  such that  $u_2(a_1^n, a_2') > u_2(a_1^n, b_2(a_1^n))$ . Define

$$b_2'(a_1) = \begin{cases} a_2', & a_1 = a_1^n \\ b_2(a_1), & a_1 <_1 a_1^n \end{cases}$$

Note that  $b_2'$  is increasing and hence incentive compatible. Also,

$$\begin{aligned} u_2(a_1^n, b_2'(a_1^n)) - u_2(a_1^n, b_2(a_1^n)) - [u_2(a_1^{n-1}, b_2'(a_1^n)) - u_2(a_1^{n-1}, b_2(a_1^n))] > 0 \\ = \sum_{j=k}^{n-1} u_2(a_1^j, b_2'(a_1^j)) - u_2(a_1^j, b_2(a_1^j)) + \sum_{j=k}^{n-2} u_2(a_1^j, b_2(a_1^{j+1})) - u_2(a_1^j, b_2'(a_1^{j+1})) \end{aligned}$$

for all  $k < n$ , which, by Proposition 3, contradicts that  $b_2$  is renegotiation-proof.  $\square$

Suppose, for contradiction, that  $u_1(a_1^*, br_2(a_1^*)) < u_1(\bar{a}_1, br_2(\bar{a}_1))$ . Then

$$u_1(\bar{a}_1, br_2(\bar{a}_1)) \geq u_1(\bar{a}_1, br_2(\bar{a}_1)) > u_1(a_1^*, br_2(a_1^*)) = u_1(a_1^*, b_2(a_1^*)),$$

where the first inequality follows from no right deviation at  $\bar{a}_1$  (Lemma 7) and positive externality. Therefore,  $u_1(\bar{a}_1, br_2(\bar{a}_1)) > u_1(a_1^*, b_2(a_1^*))$ , which contradicts that  $(a_1^*, b_2)$  is a Nash equilibrium of  $G$ , and the proof is completed.  $\square$

*Proof of Theorem 4.* By definition  $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1}$  is not strongly renegotiation-proof if and only if there exist  $i = 1, 2, \dots, n$  and incentive compatible  $(g, b_2) \in \mathcal{C} \times A_2^{A_1}$  such that  $u_2(a_1^i, b_2(a_1^i)) - g(b_2(a_1^i)) > u_2(a_1^i, b_2^*(a_1^i)) - f(b_2^*(a_1^i))$ ,  $g(b_2(a_1^i)) > f(b_2^*(a_1^i))$ , and  $g(b_2(a_1^j)) - f(b_2^*(a_1^j)) > \min\{0, u_2(a_1^j, b_2(a_1^j)) - u_2(a_1^j, b_2^*(a_1^j))\}$  for all  $j = 1, 2, \dots, n$ . The following lemma easily follows.

**Lemma 8.**  $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1}$  is not strongly renegotiation-proof if and only if there exist  $i = 1, 2, \dots, n$ ,  $b_2 \in A_2^{A_1}$ , and  $\varepsilon \in \mathbb{R}^n$  such that  $D(f(b_2^*) + \varepsilon) \leq U(b_2)$ ,  $0 < \varepsilon_i < u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i))$ , and  $\varepsilon_j > \min\{0, u_2(a_1^j, b_2(a_1^j)) - u_2(a_1^j, b_2^*(a_1^j))\}$  for all  $j = 1, 2, \dots, n$ .

Define the matrices  $V$  and  $C$  as in the proof of Theorem 2, and define the matrix  $A$  as follows: its row 1 is  $e_1$ , row  $n+2$  is  $l_i$ , and row  $j+1$ , for  $j = 1, \dots, n$ , is given by  $-\min\{0, u_2(a_1^j, b_2(a_1^j)) - u_2(a_1^j, b_2^*(a_1^j))\}e_1 + e_{j+1}$ . We have the following lemma, whose proof is similar to that of Lemma 4.

**Lemma 9.**  $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1}$  is strongly renegotiation-proof if and only if for any  $i = 1, 2, \dots, n$  and  $b_2 \in A_2^{A_1}$  there exist  $y \in \mathbb{R}^{n+2}$  and  $z \in \mathbb{R}^{2(n-1)}$  such that  $A'y + C'z = 0$ ,  $y > 0$ ,  $z \geq 0$ .

The rest of the proof is almost identical to that of Theorem 2, and therefore is omitted.  $\square$

**Lemma 10.** If  $u_1$  and  $u_2$  have strictly increasing differences, then  $NE(S(G))$  is stable.

*Proof of Lemma 10.* Assume that  $u_1$  and  $u_2$  have strictly increasing differences and fix a selection  $(br_1, br_2)$ . It is a standard result that  $br_1$  and  $br_2$  are increasing. Suppose, for contradiction, that there exists an  $a_1 \succ_1 \bar{a}_1^{NE}$  such that  $br_1(br_2(a_1)) \succ_1 a_1$ . Consider the sequence  $(a_1^t, a_2^t)$ ,  $t = 0, 1, \dots$  defined by  $a_1^0 = a_1$ ,  $a_2^t = br_2(a_1^t)$ ,  $t = 0, 1, \dots$ , and  $a_1^t = br_1(a_2^{t-1})$ ,  $t = 1, 2, \dots$

We claim that there exist  $i = 1, 2$  and  $k = 1, 2, \dots$  such that  $a_i^k = \bar{a}_i$ . We will show that if  $a_i^t <_i \bar{a}_i$  for  $i = 1, 2$  and  $t = 0, 1, \dots, k-1$ , then  $a_i^k \succ_i a_i^{k-1}$ ,  $k = 1, 2, \dots$ . The claim then follows from the

finiteness of  $A_i$ . Note that  $a_1^0 <_1 \bar{a}_1$  by assumption and assume that  $a_2^0 <_2 \bar{a}_2$ . We then have  $a_1^1 = br_1(a_2^0) = br_1(br_2(a_1^0)) >_1 a_1^0$ . Since  $br_2$  is increasing,  $a_2^1 = br_2(a_1^1) \succ_2 br_2(a_1^0) = a_2^0$ . If  $a_2^1 = a_2^0$ , then  $a_1^1 = br_1(a_2^0) = br_1(a_2^1) = a_1^2$ , which implies that  $(a_1^1, a_2^1) \in NE(S(G))$ , contradicting that  $\bar{a}_1^{NE}$  is the greatest Nash equilibrium action. Therefore, we must have  $a_2^1 >_2 a_2^0$ . This shows that the claim holds for  $k = 1$ . Now suppose that it holds for  $k = 1, 2, \dots, l-1$  and assume that  $a_i^k <_i \bar{a}_i$ , for  $i = 1, 2$  and  $k = 1, \dots, l-1$ . We then have  $a_1^l = br_1(a_2^{l-1}) \succ_1 br_1(a_2^{l-2}) = a_1^{l-1}$ . If  $a_1^l = a_1^{l-1}$ , then  $a_1^{l-1} = br_1(a_2^{l-1})$  and  $a_2^{l-1} = br_2(a_1^{l-1})$ , which implies that  $(a_1^{l-1}, a_2^{l-1}) \in NE(S(G))$ , contradicting that  $\bar{a}_1^{NE}$  is the greatest Nash equilibrium action. Therefore,  $a_1^l >_1 a_1^{l-1}$ . Similarly,  $a_2^l = br_2(a_1^l) \succ_2 br_2(a_1^{l-1}) = a_2^{l-1}$ . If  $a_2^l = a_2^{l-1}$ , then  $(a_1^l, a_2^{l-1}) \in NE(S(G))$ , again contradicting that  $\bar{a}_1^{NE}$  is the greatest Nash equilibrium action. Therefore,  $a_2^l >_2 a_2^{l-1}$ , completing the proof.

Now, assume, without loss of generality, that  $a_2^k = \bar{a}_2$  for some  $k = 1, 2, \dots$ . Then,  $a_1^{k+1} = br_1(a_2^k) \succ_1 br_1(a_2^{k-1}) = a_1^k$ . If  $a_1^{k+1} = a_1^k$ , then  $(a_1^k, a_2^k) \in NE(S(G))$ , contradicting that  $\bar{a}_1^{NE}$  is the greatest Nash equilibrium action. If  $a_1^{k+1} >_1 a_1^k$ ,  $a_2^{k+1} = br_2(a_1^{k+1}) \succ_2 br_2(a_1^k) = \bar{a}_2$ , and hence  $a_2^{k+1} = \bar{a}_2$ . This implies that  $(a_1^{k+1}, \bar{a}_2) \in NE(S(G))$ , again contradicting that  $\bar{a}_1^{NE}$  is the greatest Nash equilibrium action.  $\square$

## References

- [1] Beaudry, P. and M. Poitevin (1995), "Contract Renegotiation: A Simple Framework and Implications for Organization Theory," *Canadian Journal of Economics*, 28, 302-335.
- [2] Bensaid, B. and R. J. Gary-Bobo (1993), "Commitment Value of Contracts under Renegotiation Constraints," *Econometrica*, 61, 1423-29.
- [3] Bhaskar, V. (2009), "Games Played in a Contracting Environment," *Games and Economic Behavior*, 67, 389-398.
- [4] Bolton, P. and M. Dewatripont (2005), *Contract Theory*, Cambridge: The MIT Press.
- [5] Bolton, P. and D. S. Scharfstein (1990), "A Theory of Predation Based on Agency Problems in Financial Contracting," *American Economic Review*, 80, 93-106.
- [6] Brander, J. A. and T. R. Lewis (1986), "Oligopoly and Financial Structure: The Limited Liability Effect," *American Economic Review*, 76, 956-970.
- [7] Brander, J. A. and B. J. Spencer (1985), "Export Subsidies and International Market Share Rivalry," *Journal of International Economics*, 18, 83-100.
- [8] Caillaud, B., B. Jullien, and P. Picard (1995), "Competing Vertical Structures: Precommitment and Renegotiation," *Econometrica*, 63, 621-646.
- [9] Cho, I. K. and D. Kreps (1987), "Signaling Games and Stable Equilibria," *Quarterly Journal of Economics*, 102, 179-221.
- [10] Dewatripont, M. (1988), "Commitment through Renegotiation-Proof Contracts with Third Parties," *Review of Economic Studies*, 55, 377-389.

- [11] Eaton, J. and G. M. Grossman (1986), "Optimal Trade and Industrial Policy under Oligopoly," *Quarterly Journal of Economics*, 101, 383-406.
- [12] Fershtman, C. and K. Judd (1987), "Equilibrium Incentives in Oligopoly," *American Economic Review*, 77, 927-940.
- [13] Fershtman, C., K. Judd, and E. Kalai (1991), "Observable Contracts: Strategic Delegation and Cooperation," *International Economic Review*, 32, 551-559.
- [14] Fershtman, C. and E. Kalai (1997), "Unobserved Delegation," *International Economic Review*, 38, 763-774.
- [15] Fudenberg, D. and J. Tirole (1991), "Perfect Bayesian and Sequential Equilibrium," *Journal of Economic Theory*, 53, 236-260.
- [16] Jackson, M. O. and S. Wilkie (2005), "Endogenous Games and Mechanisms: Side Payments Among Players," *Review of Economic Studies*, 72, 543-566.
- [17] Katz, M. (1991), "Game-Playing Agents: Unobservable Contracts as Precommitments," *Rand Journal of Economics*, 22, 307-328.
- [18] Katz, M. (2006), "Observable Contracts as Commitments: Interdependent Contracts and Moral Hazard," *Journal of Economics & Management Strategy*, 15, 685-706.
- [19] Koçkesen, L. (2007), "Unobservable Contracts as Precommitments," *Economic Theory*, 31, 539-552.
- [20] Koçkesen, L. and E. A. Ok (2004), "Strategic Delegation by Unobservable Incentive Contracts," *Review of Economic Studies*, 71, 397-424.
- [21] Koçkesen, L., E. A. Ok, and R. Sethi (2000), "The Strategic Advantage of Negatively Interdependent Preferences," *Journal of Economic Theory*, 92, 274-299.
- [22] Mangasarian, O. L. (1994) *Nonlinear Programming*, New York: McGraw-Hill.
- [23] Maskin, E. and J. Tirole (1992), "The Principal-Agent Relationship with an Informed Principal, II: Common Values," *Econometrica*, 60, 1-42.
- [24] Polo, M. and P. Tedeschi (2000), "Delegation Games and Side-Contracting," *Research in Economics*, 54, 101-116.
- [25] Prat, A. and A. Rustichini (2003), "Games Played through Agents," *Econometrica*, 71, 989-1026.
- [26] Schelling, T. (1960), *The Strategy of Conflict*, Cambridge: Harvard University Press.
- [27] Sklivas, S. D. (1987), "The Strategic Choice of Managerial Incentives," *Rand Journal of Economics*, 18, 452-458.
- [28] Snyder, C. M. (1996), "Negotiation and Renegotiation of Optimal Financial Contracts Under the Threat of Predation," *Journal of Industrial Economics*, 44, 325-343.

- [29] Spencer, B. J. and J. A. Brander (1983), "International R & D Rivalry and Industrial Strategy," *Review of Economic Studies*, 50, 707-722.
- [30] Topkis, D. (1979), "Equilibrium Points in Nonzero-sum N-person Submodular Games," *SIAM Journal on Control and Optimization*, 17, 773-787.
- [31] Vickers, J. (1985), "Delegation and the Theory of the Firm," *Economic Journal, Supplement*, 95, 138-147.