1. (100pts.) A risk neutral landowner is going to offer a tenancy contract to a financially constrained tenant. The crop yield is either high, \( q_h \), or low, \( q_l \), with \( q_h > q_l \geq 0 \). The tenant chooses to put in either high or low effort. If he puts in high effort, then the probability of high output is \( p_h \), and if he chooses low effort, probability of high output is \( p_l \), with \( p_h > p_l > 0 \). The landlord cannot observe the tenant’s effort but observes whether the output is low or high, and hence feasible contracts are transfers to the tenant in case of high and low outputs, i.e., \((t_h, t_l)\). Therefore, the landlord's payoff is \( q_h - t_h \), if output is high, and \( q_l - t_l \), if output is low. If the tenant receives a transfer \( t \) and puts in high effort, then his payoff is given by \( t - c \), where \( c > 0 \) is the cost of high effort. If he chooses low effort, then his payoff is simply \( t \).

The tenant has limited liability so that \( t_l \geq 0 \) and \( t_h \geq 0 \). Also assume that the reservation utility (i.e., the payoff he gets if he does not accept the contract offer) of the tenant is zero.

(a) (30pts.) What is the optimal contract that induces high effort?

Solution

The landlord’s problem is

\[
\max_{t_l, t_h} \quad p_h(q_h - t_h) + (1 - p_h)(q_l - t_l)
\]

subject to

\[
\begin{align*}
p_h t_h + (1 - p_h) t_l - c & \geq 0 \quad (IR) \\
p_h t_h + (1 - p_h) t_l - c & \geq p_l t_h + (1 - p_l) t_l \quad (IC) \\
t_l & \geq 0 \quad (LL_l) \\
t_h & \geq 0 \quad (LL_h)
\end{align*}
\]

Let us rewrite the \( IC \) constraint as

\[
(p_h - p_l)(t_h - t_l) - c \geq 0
\]

First, \( IC \) constraint implies that \( t_h > t_l \). Therefore, \( LL_l \) implies \( LL_h \), and hence the latter can be ignored. Second, \( IR \) or \( LL_l \) must bind. Otherwise, the landlord can increase \( t_l \) and \( t_h \) by the same small amount without violating any constraint and increase profit.

First, suppose that \( IR \) binds at the solution, i.e.,

\[
t_h = \frac{c - (1 - p_h)t_l}{p_h}
\]

Therefore,

\[
(p_h - p_l)(t_h - t_l) - c = (p_h - p_l) \left( \frac{c - (1 - p_h)t_l}{p_h} - t_l \right) - c
\]

\[
= \frac{p_h - p_l}{p_h} (c - t_l) - c
\]

\[
= - \frac{p_h - p_l}{p_h} t_l - \frac{p_l}{p_h} c < 0
\]

In other words, \( IC \) constraint is violated. Therefore, \( LL_l \) must bind, i.e., \( t_l = 0 \). This, in turn, implies that \( IC \) must bind. Otherwise, \( IR \) does not bind either and the landlord can decrease \( t_h \) by some small amount and increase profit.

Therefore, the optimal contract is

\[
t_l = 0, \quad t_h = \frac{c}{p_h - p_l}
\]

(b) (20pts.) What is the optimal contract that induces low effort?

Solution
The landlord’s problem is
\[
\max_{t_l, t_h} p_h(q_h - t_h) + (1 - p_h)(q_l - t_l)
\]
subject to
\[
\begin{align*}
pt_h + (1 - p_l)t_l & \geq 0 \quad (IR) \\
p_l t_h + (1 - p_l)t_l & \geq p_h t_h + (1 - p_h)t_l - c \quad (IC) \\
t_l & \geq 0 \quad (LL_l) \\
t_h & \geq 0 \quad (LL_h)
\end{align*}
\]

It can be easily shown that \(t_l = t_h = 0\). Indeed, if \(t_h > 0\), then we can reduce it by some small amount without violating any constraints. Therefore, \(t_h = 0\). This implies that, if \(t_l > 0\), then we can reduce it by some small amount without violating any constraints, and hence \(t_l = 0\).

Optimal contract is
\[
t_l = t_h = 0
\]

(c) (10pts.) What effort level is optimal to induce?

Solution

Expected profit under high effort is
\[
p_h \left( q_h - \frac{c}{p_h - p_l} \right) + (1 - p_h)q_l
\]
and under low effort it is
\[
p_l q_h + (1 - p_h)q_l
\]

Expected profit under high effort is higher if
\[
(p_h - p_l)(q_h - q_l) \geq \frac{p_h}{p_h - p_l}c
\]

If this condition holds, it is optimal to induce high effort, otherwise it is optimal to induce low effort.

(d) (20pts.) Assume that it is optimal to induce high effort, otherwise it is optimal to induce low effort.

Solution

If the landlord monitors, then the only constraint that matters is the IR constraint, since she could punish the tenant if he puts in an effort level that she does not desire to induce. Therefore, if she wants to induce high effort, her problem is
\[
\max_{t_l, t_h} p_h(q_h - t_h) + (1 - p_h)(q_l - t_l) - w
\]
subject to
\[
\begin{align*}
pt_h + (1 - p_l)t_l - c & \geq 0 \quad (IR) \\
t_l & \geq 0 \quad (LL_l) \\
t_h & \geq 0 \quad (LL_h)
\end{align*}
\]

At the solution \(IR\) constraint must bind. If it doesn’t, then \(t_l > 0\) or \(t_h > 0\), and hence the landlord can decrease \(t_l\) or \(t_h\) without violating any of the constraints and increase profit. Therefore,
\[
p_h t_h + (1 - p_h)t_l - c = 0
\]

Substituting into the objective function, expected profit is
\[
p_h q_h + (1 - p_h)q_l - c - w
\]

Therefore, monitoring is optimal if and only if
\[
p_h q_h + (1 - p_h)q_l - c - w \geq p_h \left( q_h - \frac{c}{p_h - p_l} \right) + (1 - p_h)q_l
\]

This is equivalent to
\[
w \leq \frac{p_l}{p_h - p_l}c
\]
(e) **(20pts.)** Now suppose that there is a functioning credit market so that the there are no limited liability constraints any more. Find the landlord’s expected profit under the optimal contract and compare it with her expected profit from part (a).

**Solution**

Landlord’s problem now is

$$\max_{t_l, t_h} p_h (q_h - t_h) + (1 - p_h)(q_l - t_l)$$

subject to

$$p_h t_h + (1 - p_h)t_l - c \geq 0 \quad (IR)$$

$$(p_h - p_l)(t_h - t_l) - c \geq 0 \quad (IC)$$

At the solution the IR constraint must bind, since otherwise, the landlord could decrease $t_l$ and $t_h$ by the same small amount and increase her expected profit. Therefore,

$$p_h t_h + (1 - p_h)t_l - c = 0$$

Substituting into the expected profit function we get

$$p_h q_h + (1 - p_h)q_l - c$$

Expected profit of the landlord when the tenant is financially constrained has been found as

$$p_h q_h + (1 - p_h)q_l - \frac{p_h}{p_h - p_l} c$$

Since

$$\frac{p_h}{p_h - p_l} > 1$$

expected profit when the tenant is not financially constrained is higher.