Game Theory
Solutions to Problem Set 2

1. **Question 1:** The set of Nash equilibria = \{(0, 0), (0.25, 0.25), (0.5, 0.5), (0.75, 0.75), (1, 1), (0, 1)\}. Note that these are not payoffs but action profiles.

   **Question 2:** The set of Nash equilibria = \{(U, M), (D, L), (D, M)\}.

   **Question 3:** Case I, The set of Nash equilibria = \{(200, 200), (300, 300), (400, 400)\}.

   **Question 3:** Case II, the following outcomes are Nash equilibria

   \(\text{Case I}, \text{The set of Nash equilibria} = \{(0, 0), (1, 1)\}\).

   (b) Let’s check out payoff profiles when \(\alpha \geq 0\) is a variable.

   \[
   \begin{align*}
   u_1(D, D) &= 1 + 1 \cdot \alpha = 1 + \alpha, \quad u_1(C, D) = 0 + 3 \cdot \alpha = 3\alpha, \\
   u_1(D, C) &= 3 + 0 \cdot \alpha = 3, \quad u_1(C, C) = 2 + 2 \cdot \alpha = 2 + 2\alpha. \\
   u_2(D, D) &= 1 + 1 \cdot \alpha = 1 + \alpha, \quad u_2(C, D) = 3 + 0 \cdot \alpha = 3, \\
   u_2(D, C) &= 0 + 3 \cdot \alpha = 3\alpha, \quad u_2(D, D) = 2 + 2 \cdot \alpha = 2 + 2\alpha.
   \end{align*}
   \]

   The bimatrix representation is,

   \[
   \begin{array}{cc}
   D & C \\
   \hline
   D & 2 + 2\alpha & 3 + 3\alpha \\
   C & 3\alpha & 2 + 2\alpha
   \end{array}
   \]

   This is not a Prisoners’ Dilemma game because cooperate \(C\) is dominant strategy for each player and the best outcome for both can be achieved without “dilemma.”

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   \begin{align*}
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   u_2(D, D) &= 1 + 1 \cdot \alpha = 1 + \alpha, \quad u_2(C, D) = 3 + 0 \cdot \alpha = 3, \\
   u_2(D, C) &= 0 + 3 \cdot \alpha = 3\alpha, \quad u_2(D, D) = 2 + 2 \cdot \alpha = 2 + 2\alpha.
   \end{align*}
   \]

   The bimatrix representation is,

   \[
   \begin{array}{cc}
   D & C \\
   \hline
   D & 1 + \alpha & 3 \alpha \\
   C & 3\alpha & 2 + 2\alpha
   \end{array}
   \]

   In the Prisoners’ Dilemma game, \((D, D)\) must be a dominant strategy equilibrium. Therefore, it must be that

   \[
   \begin{align*}
   u_1(D, D) &> u_1(C, D) \iff 1 + \alpha > 3\alpha \\
   u_1(D, C) &> u_1(C, C) \iff 3 > 2 + 2\alpha \\
   u_2(D, D) &> u_2(D, C) \iff 1 + \alpha > 3\alpha \\
   u_2(C, C) &> u_2(C, D) \iff 3 > 2 + 2\alpha
   \end{align*}
   \]

   These inequalities can be reduced to, \(0 \leq \alpha < \frac{1}{2}\). Note that for all \(\alpha \geq 0\) we have \(u_i(C, C) > u_i(D, D)\) which is another requirement for Prisoners’ dilemma.

   For values of \(\alpha\) such that \(0 \leq \alpha < \frac{1}{2}\), the resulting game is the Prisoners’ Dilemma.

   For \(\alpha \geq \frac{1}{2}\), the game is not.

   In both cases, there are dominant strategy equilibrium which is also a Nash equilibrium. For \(0 \leq \alpha < \frac{1}{2}\), it is \((D, D)\). For \(\frac{1}{2} < \alpha\), it is \((C, C)\). \(\alpha = \frac{1}{2}\) is a special case where all four outcomes, \((D, D), (C, D), (D, C), (C, C)\), are Nash equilibria.
3. This question relates to the private provision of public goods problem. It is also called a tragedy of commons because public goods tend to be underprovided. A good example is the cooperation levels by each individuals for a study group to solve the problem sets. People tend to make less effort when the outcome depends on the whole group’s effort level while individual preparation is costly to themselves.

(a) This situation can be expressed as the following strategic form game: players, \( N = \{1, 2\} \), action space \( A_1 = A_2 = \{ x_i \in [0, 1] : i = 1, 2 \} \), and the payoff functions are specified by the net surplus from the project, \( u_1 (x_1, x_2) = \frac{1}{2} f(x_1, x_2) - c(x_1), \) \( u_2 (x_1, x_2) = \frac{1}{2} f(x_1, x_2) - c(x_2) \).

(b) (i) \( u_1 (x_1, x_2) = \frac{1}{2} \cdot 3x_1 \cdot x_2 - (x_1)^2, \) \( u_2 (x_1, x_2) = \frac{1}{2} \cdot 3x_1 \cdot x_2 - (x_2)^2 \). First, we derive best response (reaction) function for each player given the other players strategy. Because the problem is symmetric, I will show only for player 1 first and then apply the result to the case for player 2.

\[
\max_{x_1} u_1 (x_1, x_2) = \frac{3}{2} x_1 \cdot x_2 - (x_1)^2 \text{ subject to } x_i \in [0, 1]
\]

since the function is concave in \( x_1 \). First Order Condition will give us \( x_1 = B_1 (x_2) = \frac{3}{4} x_2 \), where \( B_1 (\cdot) \) is the best response correspondence for player 1. Similarly, \( x_2 = B_2 (x_1) = \frac{1}{4} x_1 \). The solution to these two equations with two unknowns are \( x_1^* = x_2^* = 0 \). This is a Nash equilibrium. The payoffs from Nash equilibrium play are \( u_i (x_1^*, x_2^*) = u_i (0, 0) = 0 \) for \( i = 1, 2 \). You can now see why the group member did not prepare for the study group before they meet. (If \( x_1, x_2 \) are the preparation by individuals for the study group and \( f(x_1, x_2) \) is the outcome from the study group when \( c(x_1), c(x_2) \) are the cost for the individual to prepare.)

(ii) \( u_1 (x_1, x_2) = \frac{1}{2} 4x_1 \cdot x_2 - x_1, \) \( u_2 (x_1, x_2) = \frac{1}{2} 4x_1 \cdot x_2 - x_2 \). First, we derive best response correspondence for each player given the conjecture about the other players strategy. Because the problem is symmetric, I will show only for player 1.

\[
\max_{x_1} u_1 (x_1, x_2) = 2x_1 \cdot x_2 - x_1 = (2x_2 - 1)x_1 \text{ subject to } x_i \in [0, 1]
\]

The payoff function is linear in \( x_1 \). Therefore, we cannot use calculus to solve the problem. However, the domain of the function is a closed interval, hence a compact set. Therefore, depending on the slope of the linear function, we can pick the end points. For \( x_2 > \frac{1}{2} \), then the slope is strictly positive. Therefore, the value of the linear function is maximized at the highest value of \( x_1 \). Thus, \( x_1 = B_1 (x_2) = 1 \), for \( 1 > x_2 > \frac{1}{2} \). On the other hand, if \( x_2 < \frac{1}{2} \), then the slope is strictly negative. Therefore, the value of the linear function is maximized at the lowest value of \( x_1 \). Thus, \( x_1 = B_1 (x_2) = 0 \), for \( 0 \leq x_2 < \frac{1}{2} \). When \( x_2 = \frac{1}{2} \) the payoff function becomes \( u_1 (x_1, x_2) = 0 \cdot x_1 = 0 \). Therefore, no matter what the player 1 does, his payoff will not change and hence \( B_1 (x_2) = [0, 1] \) for \( x_2 = \frac{1}{2} \). To summarize

\[
B_1 (x_2) = \begin{cases} 
\{ 1 \}, & 0 \leq x_2 < \frac{1}{2} \\
[0, 1], & x_2 = \frac{1}{2} \\
\{ 0 \}, & \frac{1}{2} \leq x_2 \leq 1
\end{cases}
\]

and similarly,

\[
B_2 (x_1) = \begin{cases} 
\{ 1 \}, & 0 \leq x_1 < \frac{1}{2} \\
[0, 1], & x_1 = \frac{1}{2} \\
\{ 0 \}, & \frac{1}{2} \leq x_1 \leq 1
\end{cases}
\]
we easily see that there are 3 Nash equilibria: $(0,0), (\frac{1}{2}, \frac{1}{2}), (1,1)$. Payoffs from these equilibria are $u_i(0,0) = 0, u_i(\frac{1}{2}, \frac{1}{2}) = 0, u_i(1,1) = 1$ for $i = 1, 2$. This now has a coordination outcome. If the other guy is non-cooperative, then you end up happier if you do not cooperate. However, if the other member is cooperative, then you might as well cooperate.

(c) (i) If both players can play maximum effort level, such that $x_1^{**} = x_2^{**} = 1$, then this combination of play will give better payoffs to both of them $u_i(x_1^{**}, x_2^{**}) = u_i(1,1) = \frac{1}{2}$. remember, the Nash payoffs were $u_i(x_1^*, x_2^*) = u_i(0,0) = 0$ for $i = 1, 2$.

(ii) In this multiple equilibrium case, one Nash equilibrium gives highest payoffs to both players. That is the outcome: $(x_1^*, x_2^*) = (1,1)$. Payoffs: $u_i(1,1) = 1$ for $i = 1, 2$. 

Plotting the best response correspondences