1. Assuming that in the case of a tie, the object goes to person 1, the best response correspondences for a two person first price auction are:

\[
B_1(b_2) = \begin{cases} 
\{b_2\}, b_2 < v_1 \\
[0,b_2], b_2 = v_1 \\
[0,b_2), b_2 > v_1 
\end{cases} \quad B_2(b_1) = \begin{cases} 
\text{undefined}, b_1 < v_2 \\
[0,b_1], b_1 = v_2 \\
[0,b_1], b_1 > v_2 
\end{cases}
\]

Graphed, this looks like (where the areas with horizontal lines and the dark line segment are \(B_1\), and the area with vertical lines is \(B_2\)):

![Graph of best response correspondences](image)

Note that \(B_1\) is an open set when it is strictly greater than \(v_1\). Therefore, the only points where these two sets overlap are on the stretch between \(v_1\) and \(v_2\). The set of Nash Equilibria is then \(\{(b_1^*, b_2^*) : v_2 \leq b_1^* = b_2^* \leq v_1\}\).
2. Assuming that in the case of a tie, the object goes to person 1, the best response correspondences for a two person second price auction are:

\[ B_1(b_2) = \begin{cases} [b_2, \infty), b_2 < v_1 \\ [0, \infty), b_2 = v_1 \\ (0, b_2), b_2 > v_1 \end{cases} \]

\[ B_2(b_1) = \begin{cases} (b_1, \infty), b_1 < v_2 \\ [0, \infty), b_1 = v_2 \\ [0, b_1], b_1 > v_2 \end{cases} \]

Graphed, this looks like (where the areas with horizontal lines and the dark line segment are \( B_1 \) and the areas with vertical lines and the lighter line segment are \( B_2 \). The areas where the two correspondences intersect are marked by cross hatches.):

Note that \( B_1 \) is an open set when it is strictly greater than \( v_1 \), while \( B_1 \) is an open set when it is strictly less than \( v_2 \). Therefore, the boundaries of the correspondences only intersect along the \( b_1=b_2 \) line segment from \( v_1 \) to \( v_2 \).

\[ \text{NE} = \{(b_1^*, \ldots, b_n^*) : b_1^* \leq v_2, v_1 \leq b_2^* \} \cup \{(b_1^*, b_2^*) : b_1^* \geq v_2, v_1 \geq b_2^*, b_1^* \geq b_2^* \} \].

3. a) A vector of bids \((b_1^*, \ldots, b_n^*)\) is a Nash Equilibrium of the modified first price auction if and only if it satisfies:

i. \( b_i^* \leq b_1^* \) for all \( i \neq 1 \),

ii. \( v_2 \leq b_1^* \leq v_1 \), and

iii. \( b_1^* = b_S^* \), where \( b_S^* = \max \{ b_2^*, \ldots, b_n^* \} \).

Proof (if direction): Any vector that satisfies these conditions is a Nash Equilibrium.

Suppose there is a bid vector \((b_1^*, \ldots, b_n^*)\) that satisfies i. and ii.

1 has no profitable deviation. He has the highest bid (by condition i.) and has won the object, paying his bid, and receiving a payoff of \( v_1 - b_1^* \geq 0 \) (by condition ii.). Changing his bid to any \( b_1 \geq b_1^* \) will only increase the amount he must pay and so decrease his
payoff (since he has already won with $b_1^*$). Changing his bid to any $b_1' < b_1^* = b_S^*$ (by condition iii.) will result in $b_S^*$ becoming the highest bid and whoever bid it winning the object, and so 1’s payoff will decrease to 0.

No other player has a profitable deviation, either. Currently player i for all $i \neq 1$ is neither winning the object nor paying anything, and so her payoff is 0. Deviating to any $b_i' \leq b_1^*$ will not change the outcome for her and so will not change her payoff. And deviating to any $b_i' > b_1^*$ will result in her winning the object and having to pay $b_i'^* > b_1'^* \geq v_2 > v_i$, and so her payoff will decrease to $b_1'^* - v_i < 0$.

No player has a profitable deviation, so $(b_1^*, \ldots, b_n^*)$ is a Nash Equilibrium.

(only if direction): Any Nash Equilibrium vector must satisfy the above conditions.

I) Proof that i. is a necessary condition.

Let $(b_1', \ldots, b_n')$ be a vector for which condition i. does not hold. That is, suppose that there is some $b_i' > b_1'$, so that player i wins the object instead of player 1, giving 1 a payoff of 0 and i a payoff of $v_i - b_i'$. $(b_1', \ldots, b_n')$ cannot then be a Nash Equilibrium, because either player 1 or player i has a profitable deviation.

If $b_i' \leq v_i < v_1$, then 1 can profitably deviate to playing $b_1'' = b_i'$. Then 1 would have the weakly highest bid, and so by the rules of the modified auction 1 would win the auction and receive a payoff of $v_1 - b_1'' = v_1 - b_i' > 0$.

If $b_i' > v_n$, then i’s payoff is $v_i - b_i' < 0$. i can profitably deviate to playing, e.g., $b_i'' = 0$, so that some other player will win and i’s payoff will increase to 0.

II) Proof that ii. is a necessary condition.

Let $(b_1', \ldots, b_n')$ be a vector for which condition i. holds, but condition ii. does not. That is, 1 has won with a bid $b_1'$ such that either $b_1' > v_1$ or $b_1' < v_2$. Then $(b_1', \ldots, b_n')$ cannot be a Nash Equilibrium, because either player 1 or player 2 will have a profitable deviation.

If $b_1' > v_1$, then 1’s payoff is $v_1 - b_1' < 0$. Like i in the previous step, 1 can profitably deviate to playing, e.g., $b_1'' = 0$, so that some other player will win and 1’s payoff will increase to 0.

If $b_1' < v_2$, then by completeness of the real numbers, there is some $b_2''$ such that $b_1' < b_2'' < v_2$. 2 can profitably deviate to $b_2''$, win the object, and receive a payoff of $v_2 - b_2'' > 0$, which is an improvement over the payoff of 0 2 gets by playing $b_2'$ and letting 1 win.
III) Proof that iii. is a necessary condition.

Let \((b_1', \ldots , b_n')\) be a vector for which conditions i. and ii. hold, but condition iii. does not. That is \(b_1 > b_S \geq b_i\) for all \(i>1\). Then \((b_1', \ldots , b_n')\) cannot be a Nash Equilibrium, because 1 can profitably deviate to \(b_1'' = b_S'\). He will still win the object, and his payoff will increase to \(v_1 - b_i'' > v_1 - b_i'\).

This concludes our proof of 3.a).

b) The unique weakly dominant strategy equilibrium for the modified second price auction is for all players to bid \((b_1, \ldots b_n) = (v_1, \ldots v_n)\).

Proof: \(b_i = v_i\) is a weakly dominant strategy for all \(i\).

Fix \(b_i \epsilon A_{i,j}\). Let \(b_H\) be the maximum bid in \(b_{-i}\), and let \(h\) be the index of the player who has bid \(b_H\).

If \(b_H > v_i\), player \(i\) receives a payoff of 0 from making any non-winning bid, that is \(u_i(b_i, b_{-i}) = 0\) for any \(b_i \epsilon [0, b_H]\) if \(i < h\), or any \(b_i \epsilon [0, b_H)\) if \(i > h\). (Note that \(v_i\) is an element of these non-winning intervals.) On the other hand, if \(i\) makes a winning bid, she receives a negative payoff; that is, \(u_i(b_i, b_{-i}) = v_i - b_H < 0\) for any \(b_i \epsilon (b_H, \infty)\) if \(i < h\), or any \(b_i \epsilon [b_H, \infty)\) if \(i > h\). Thus for this case, playing any non-winning bid (including \(v_i\)) is no worse than playing any other non-winning bid, and is strictly better than playing any winning bid.

On the other hand, if \(b_H < v_i\), player \(i\) still receives a payoff of 0 from making any non-winning bid, that is \(u_i(b_i, b_{-i}) = 0\) for any \(b_i \epsilon [0, b_H]\) if \(i < h\), or any \(b_i \epsilon [0, b_H)\) if \(i > h\). On the other hand, if \(i\) makes a winning bid, she now receives a positive payoff; that is, \(u_i(b_i, b_{-i}) = v_i - b_H > 0\) for any \(b_i \epsilon (b_H, \infty)\) if \(i < h\), or any \(b_i \epsilon [b_H, \infty)\) if \(i > h\). (Note that this time, \(v_i\) is an element of the winning interval.) Thus for this case, playing any winning bid (including \(v_i\)) is no worse than playing any other winning bid, and is strictly better than playing any non-winning bid.

Lastly, if \(b_H = v_i\), player \(i\) receives a payoff of 0 from making any bid whatever. That is, \(u_i(b_i, b_{-i}) = 0\) for any \(b_i \epsilon [0, b_H]\) if \(i < h\), or any \(b_i \epsilon [0, b_H)\) if \(i > h\) (that is, for any non-winning bid. On the other hand, \(u_i(b_i, b_{-i}) = v_i - b_H = 0\) for any \(b_i \epsilon (b_H, \infty)\) if \(i < h\), or any \(b_i \epsilon [b_H, \infty)\) if \(i > h\) (that is, for any winning bid). Thus for this case, playing any bid (including \(v_i\)) is no worse than playing any other bid.

Thus we see that playing \(b_i = v_i\) is a weakly dominant strategy, since it is never worse than playing any other strategy and is sometimes strictly better. Since this is true for all \(i \epsilon N\), every player has a weakly dominant strategy, and so \((b_1, \ldots b_n) = (v_1, \ldots v_n)\) is a weakly dominant strategy equilibrium.
Note that no other possible bid besides \( v_i \) is an element of the winning interval when winning is preferred, and also an element of the non-winning interval when not winning is preferred. Thus there is no other bid that is a weakly dominant strategy, and hence no other weakly dominant strategy profile.

For an example of a Nash Equilibrium where 1 does not win, look at the graph for problem number 2. This is a generalized second price auction where \( n = 2 \). Note that the area of Nash Equilibria on the upper left consists entirely of equilibria where 2 wins the object. By bidding more than \( v_1 \), 2 prevents 1 from over bidding her, and by bidding less than \( v_2 \), 1 gives 2 no incentive to drop her bid.

4. \( N = \{1, 2\} \)
   \( A_i = [0, \infty) \), for all i
   \[ u_i(P_i, P_j) = \begin{cases} P_i(10 + P_j - \alpha P_j), \alpha P_i < 10 + P_j \\ 0, \text{else} \end{cases} \]

We find the Best Response Functions by setting \( \partial u_i / \partial P_i = 0 \).

\[ BR_i(P_j) = \frac{10 + P_j}{2\alpha} \]. Note that this function will be positive for any positive value \( P_j \), and so (since j’s action space is limited to positive numbers) we don’t have to worry about ensuring that \( P_i \) is positive.

Now to find the Nash Equilibrium, we set \( P_1 = BR_1(BR_2(P_1)) \), i.e. we plug the equations into each other. This gives us \( (P_1^*, P_2^*) = (10/(2\alpha-1), 10/(2\alpha-1)) \).

We find profit as a function of \( \alpha \) by plugging these prices into the profit function. This gives us \( u_i(\alpha) = \frac{100\alpha}{(2\alpha-1)^2} \). We then take the derivative of this expression with respect to \( \alpha \) and find it is negative whenever \( \alpha > 1 \), as it is defined to be in this problem. Thus, when \( \alpha \) increases, profit decreases. Intuitively, a firm with a more price sensitive group of customers cannot charge as high a price, and so makes lower profits.