1. The backward induction equilibrium of the game is \((SSS, SSS)\), i.e., each player plays \(S\) whenever it is her turn to move.

2. Given the child’s action the parent maximizes the following payoff function

\[
\min \{ c(a) + m, p(a) - m \}
\]

with respect to \(m\), i.e., the money transferred to the child. The optimal \(m\) is then found by

\[
c(a) + m^* = p(a) - m^*
\]

or

\[
m^* = \frac{p(a) - c(a)}{2}.
\]

Given this the child’s payoff function is

\[
c(a) + m^* = c(a) + \frac{p(a) - c(a)}{2} = \frac{p(a) + c(a)}{2}.
\]

Therefore, the child chooses \(a\) to maximize the sum of her and her parent’s income.

3. Assume that \(0 < \beta < 1\). The payoffs if player 2 says yes \((Y)\) to offer \(x_1\) are

\[
((1 + \beta) x_1 - \beta, 1 - (1 + \beta) x_1),
\]

whereas each receive 0 if she says no \((N)\). Therefore, player 2 will say yes if

\[
1 - (1 + \beta) x_1 > 0
\]

or

\[
x_1 < \frac{1}{1 + \beta},
\]

will say no if

\[
x_1 > \frac{1}{1 + \beta},
\]


and she is indifferent if 
\[ x_1 = \frac{1}{1+\beta}. \]

Suppose player 2’s equilibrium strategy is

\[ s_2(x_1) = \begin{cases} 
Y, & \text{if } x_1 \leq \frac{1}{1+\beta}, \\
N, & \text{if } x_1 > \frac{1}{1+\beta}.
\end{cases} \]

Given this strategy, player 1’s payoff function is

\[ u_1(x_1, s_2(x_1)) = \begin{cases} 
(1 + \beta) x_1 - \beta, & \text{if } x_1 \leq \frac{1}{1+\beta} \\
0, & \text{if } x_1 > \frac{1}{1+\beta}.
\end{cases} \]

Given that \( \beta \in (0, 1) \) we have 
\[ (1 + \beta) x_1 - \beta > 0 \]
for \( x_1 = \frac{1}{1+\beta} \), and hence player 1 will offer \( \frac{1}{1+\beta} \). Therefore,

\[ s_1^* = \frac{1}{1+\beta} \]

\[ s_2^*(x_1) = \begin{cases} 
Y, & \text{if } x_1 \leq \frac{1}{1+\beta} \\
N, & \text{if } x_1 > \frac{1}{1+\beta}.
\end{cases} \]

is a subgame perfect equilibrium. Now, suppose the equilibrium strategy of player 2 is

\[ s_2(x_1) = \begin{cases} 
Y, & \text{if } x_1 < \frac{1}{1+\beta} \\
N, & \text{if } x_1 \geq \frac{1}{1+\beta}.
\end{cases} \]

In this case player 1’s payoff function is

\[ u_1(x_1, s_2(x_1)) = \begin{cases} 
(1 + \beta) x_1 - \beta, & \text{if } x_1 < \frac{1}{1+\beta} \\
0, & \text{if } x_1 \geq \frac{1}{1+\beta}.
\end{cases} \]

which has no maximum. Therefore, the unique subgame perfect equilibrium is given by

\[ s_1^* = \frac{1}{1+\beta} \]

\[ s_2^*(x_1) = \begin{cases} 
Y, & \text{if } x_1 \leq \frac{1}{1+\beta} \\
N, & \text{if } x_1 > \frac{1}{1+\beta}.
\end{cases} \]

4. In the subgame following player 2’s offer \( x_2 \), player 1 compares \( \delta (1 - x_2) \) and \( \delta^2 y \) to decide whether to accept the offer or not. As in the usual ultimatum game there is
no equilibrium in which player 1 rejects the offer when he is indifferent, i.e., when 
\( \delta (1 - x_2) = \delta^2 y \) (verify). Therefore, player 1’s equilibrium strategy in the subgame is 

\[
s_1^*(x_2) = \begin{cases} 
Y, & \text{if } x_2 \leq 1 - \delta y \\
N, & \text{if } x_2 > 1 - \delta y 
\end{cases}
\]

Therefore, player 2’s optimal offer is 

\[
x_2^* = 1 - \delta y.
\]

Given these strategies, player 2 compares \( 1 - x_1 \) and \( \delta x_2^* = \delta (1 - \delta y) \) when faced with offer \( x_1 \). Again there is no equilibrium in which player 2 rejects when she is indifferent and hence her optimal strategy is 

\[
s_2^*(x_1) = \begin{cases} 
Y, & \text{if } x_1 \leq 1 - \delta + \delta^2 y \\
N, & \text{if } x_2 > 1 - \delta + \delta^2 y 
\end{cases}
\]

Thus, player 1’s optimal offer is 

\[
x_1^* = 1 - \delta + \delta^2 y.
\]

To sum, \((x_1^*, s_2^*(x_1), x_2^*, s_1^*(x_2))\) is the unique subgame perfect equilibrium. The SPE outcome is for player 1 to offer \( 1 - \delta + \delta^2 y \) and for player 2 accept.