Game Theory
Strategic Form Games: Applications

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Outline

1. Auctions
2. Price Competition Models
3. Elections
Auctions

Many economic transactions are conducted through auctions

- treasury bills
- foreign exchange
- publicly owned companies
- mineral rights
- airwave spectrum rights

Also can be thought of as auctions

- takeover battles
- queues
- wars of attrition
- lobbying contests

- art work
- antiques
- cars
- houses
- government contracts
Auction Formats

1. Open bid auctions
   1.1 ascending-bid auction
      ★ aka English auction
      ★ price is raised until only one bidder remains, who wins and pays the final price
   1.2 descending-bid auction
      ★ aka Dutch auction
      ★ price is lowered until someone accepts, who wins the object at the current price

2. Sealed bid auctions
   2.1 first price auction
      ★ highest bidder wins; pays her bid
   2.2 second price auction
      ★ aka Vickrey auction
      ★ highest bidder wins; pays the second highest bid
Auctions also differ with respect to the valuation of the bidders

1. Private value auctions
   - each bidder knows only her own value
   - artwork, antiques, memorabilia

2. Common value auctions
   - actual value of the object is the same for everyone
   - bidders have different private information about that value
   - oil field auctions, company takeovers
We will study sealed bid auctions

- For now we will assume that values are common knowledge
  - value of the object to player $i$ is $v_i$ dollars
- For simplicity we analyze the case with only two bidders
- Assume $v_1 > v_2 > 0$
Second Price Auctions

- Highest bidder wins and pays the second highest bid
- In case of a tie, the object is awarded to player 1

Strategic form:

1. \( N = \{1, 2\} \)
2. \( A_1 = A_2 = \mathbb{R}_+ \)
3. Payoff functions: For any \((b_1, b_2) \in \mathbb{R}_+^2\)

\[
  u_1(b_1, b_2) = \begin{cases} 
    v_1 - b_2, & \text{if } b_1 \geq b_2, \\
    0, & \text{otherwise.}
  \end{cases}
\]

\[
  u_2(b_1, b_2) = \begin{cases} 
    v_2 - b_1, & \text{if } b_2 > b_1, \\
    0, & \text{otherwise.}
  \end{cases}
\]
Second Price Auctions

I. Bidding your value weakly dominates bidding higher

Suppose your value is $10 but you bid $15. Three cases:

1. The other bid is higher than $15 (e.g. $20)
   - You lose either way: no difference

2. The other bid is lower than $10 (e.g. $5)
   - You win either way and pay $5: no difference

3. The other bid is between $10 and $15 (e.g. $12)
   - You lose with $10: zero payoff
   - You win with $15: lose $2
Second Price Auctions

II. Bidding your value weakly dominates bidding lower

Suppose your value is $10 but you bid $5. Three cases:

1. The other bid is higher than $10 (e.g. $12)
   - You loose either way: no difference

2. The other bid is lower than $5 (e.g. $2)
   - You win either way and pay $2: no difference

3. The other bid is between $5 and $10 (e.g. $8)
   - You loose with $5: zero payoff
   - You win with $10: earn $2

Weakly dominant strategy equilibrium = $(v_1, v_2)$

There are many Nash equilibria. For example $(v_1, 0)$
First Price Auctions

- Highest bidder wins and pays her own bid
- In case of a tie, the object is awarded to player 1

Strategic form:

1. $N = \{1, 2\}$
2. $A_1 = A_2 = \mathbb{R}_+$
3. Payoff functions: For any $(b_1, b_2) \in \mathbb{R}_+^2$

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_1, & \text{if } b_1 \geq b_2, \\ 0, & \text{otherwise.} \end{cases}$$

$$u_2(b_1, b_2) = \begin{cases} v_2 - b_2, & \text{if } b_2 > b_1, \\ 0, & \text{otherwise.} \end{cases}$$
There is no dominant strategy equilibrium

How about Nash equilibria?

We can compute the best response correspondences

or we can adopt a direct approach

- You first find the necessary conditions for a Nash equilibrium
  - If a strategy profile is a Nash equilibrium then it must satisfy these conditions
- Then you find the sufficient conditions
  - If a strategy profile satisfies these conditions, then it is a Nash equilibrium
Necessary Conditions

Let \((b_1^*, b_2^*)\) be a Nash equilibrium. Then,

1. Player 1 wins: \(b_1^* \geq b_2^*\)

Proof

Suppose not: \(b_1^* < b_2^*\). Two possibilities:

1.1 \(b_2^* \leq v_2\): Player 1 could bid \(v_2\) and obtain a strictly higher payoff

1.2 \(b_2^* > v_2\): Player 2 has a profitable deviation: bid zero

Contradicting the hypothesis that \((b_1^*, b_2^*)\) is a Nash equilibrium.

2. \(b_1^* = b_2^*\)

Proof

Suppose not: \(b_1^* > b_2^*\). Player 1 has a profitable deviation: bid \(b_2^*\)

3. \(v_2 \leq b_1^* \leq v_1\)

Proof

Exercise
So, any Nash equilibrium \((b_1^*, b_2^*)\) must satisfy

\[ v_2 \leq b_1^* = b_2^* \leq v_1. \]

Is any pair \((b_1^*, b_2^*)\) that satisfies these inequalities an equilibrium?

Set of Nash equilibria is given by

\[ \{ (b_1, b_2) : v_2 \leq b_1 = b_2 \leq v_1 \} \]
Price Competition Models

- Quantity (or capacity) competition: Cournot Model
  - Augustine Cournot (1838)
- Price Competition: Bertrand Model
  - Joseph Bertrand (1883)

Two main models:

1. Bertrand Oligopoly with Homogeneous Products
2. Bertrand Oligopoly with Differentiated Products
Bertrand Duopoly with Homogeneous Products

- Two firms, each with unit cost $c \geq 0$
- They choose prices
  - The one with the lower price captures the entire market
  - In case of a tie they share the market equally
- Total market demand is equal to one (not price sensitive)

Strategic form of the game:

1. $N = \{1, 2\}$
2. $A_1 = A_2 = \mathbb{R}_+$
3. Payoff functions: For any $(P_1, P_2) \in \mathbb{R}_+^2$

$$u_1(P_1, P_2) = \begin{cases} P_1 - c, & \text{if } P_1 < P_2, \\ \frac{P_1 - c}{2}, & \text{if } P_1 = P_2, \\ 0, & \text{if } P_1 > P_2. \end{cases}$$

$$u_2(P_1, P_2) = \begin{cases} P_2 - c, & \text{if } P_2 < P_1, \\ \frac{P_2 - c}{2}, & \text{if } P_2 = P_1, \\ 0, & \text{if } P_2 > P_1. \end{cases}$$
Nash Equilibrium

Suppose $P_1^*, P_2^*$ is a Nash equilibrium. Then

1. $P_1^*, P_2^* \geq c$. Why?

2. At least one charges $c$
   - $P_1^* > P_2^* > c$?
   - $P_2^* > P_1^* > c$?
   - $P_1^* = P_2^* > c$?

3. $P_2^* > P_1^* = c$?

4. $P_1^* > P_2^* = c$?

The only candidate for equilibrium is $P_1^* = P_2^* = c$, and it is indeed an equilibrium.

The unique Nash equilibrium of the Bertrand game is $(P_1^*, P_2^*) = (c, c)$

- What if unit cost of firm 1 exceeds that of firm 2?
- What if prices are discrete?
Bertrand Duopoly with Differentiated Products

- Two firms with products that are imperfect substitutes
- The demand functions are

\[ Q_1(P_1, P_2) = 10 - \alpha P_1 + P_2 \]
\[ Q_2(P_1, P_2) = 10 + P_1 - \alpha P_2 \]

- Assume that \( \alpha > 1 \)
- Unit costs are \( c \)

**Exercise**
Formulate as a strategic form game and find its Nash equilibria.
A Model of Election

Spatial Voting Models

- **Candidates** choose a policy
  - 10% tax rate vs. 25% tax rate
  - pro-EU vs anti-EU
- **Only goal is to win the election**
  - preferences: win $\succ$ tie $\succ$ lose
- **Voters** have ideal positions over the issue
  - one voter could have 15% as ideal tax rate, another 45%
- One-dimensional policy space: $[0, 1]$
- Identify each voter with her ideal position $t \in [0, 1]$
- Voters’ preferences are single peaked
  - They vote for that candidate whose position is closest to their ideal point
- Society is a continuum and voters are distributed uniformly over $[0, 1]$
Strategic Form of the Game

1. \( N = \{1, 2\} \)
2. \( A_1 = A_2 = [0, 1] \)
3. 

\[
u_i(p_1, p_2) = \begin{cases} 
1, & \text{if } i \text{ wins} \\
\frac{1}{2}, & \text{if there is a tie} \\
0, & \text{if } i \text{ loses}
\end{cases}
\]

Say the two candidates choose \( 0 < p_1 < p_2 < 1 \)

![Diagram showing the strategic form of the game with a continuous scale between 0 and 1, indicating the probabilities of winning, losing, or a tie.]
Nash Equilibrium

Suppose \( p_1^*, p_2^* \) is a Nash equilibrium. Then

1. Outcome must be a tie
   - Whatever your opponent chooses you can always guarantee a tie

2. \( p_1^* \neq p_2^* \)?

3. \( p_1^* = p_2^* \neq 1/2 \)?

The only candidate for equilibrium is \( p_1^* = p_2^* = 1/2 \), which is indeed an equilibrium.

The unique Nash equilibrium of the election game is \( (p_1^*, p_2^*) = (1/2, 1/2) \)
Other Election Models

- This result generalizes to models with more general distributions
- Equilibrium is for each party to choose the median position
  - Known as the median voter theorem

Other Models

- Models with participation costs
- Models with more than two players
- Models with multidimensional policy space
- Models with ideological candidates