Game Theory and Strategy

Introduction

Levent Koçkesen
Koç University

Game Theory: Definition and Assumptions

- Game theory studies strategic interactions within a group of individuals
  - Actions of each individual have an effect on the outcome
  - Individuals are aware of that fact
- Individuals are rational
  - have well-defined objectives over the set of possible outcomes
  - implement the best available strategy to pursue them
- Rules of the game and rationality are common knowledge

Example

10 people go to a restaurant for dinner
Order expensive or inexpensive fish?
- Expensive fish: value = 18, price = 20
- Inexpensive fish: value = 12, price = 10
Everbody pays own bill
- What do you do?
- It is a GAME

Example: A Single Person Decision Problem

Ali is an investor with $100

<table>
<thead>
<tr>
<th>State</th>
<th>Good</th>
<th>Bad</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bonds</td>
<td>10%</td>
<td>10%</td>
</tr>
<tr>
<td>Stocks</td>
<td>20%</td>
<td>0%</td>
</tr>
</tbody>
</table>

Which one is better?
- Probability of the good state $p$
- Assume that Ali wants to maximize the amount of money he has at the end of the year.
- Bonds: $110
- Stocks: average (or expected) money holdings:
  
  $p \times 120 + (1 - p) \times 100 = 100 + 20 \times p$

- If $p > 1/2$ invest in stocks
- If $p < 1/2$ invest in bonds
An Investment Game

- Ali again has two options for investing his $100:
  - invest in bonds
    - certain return of 10%
  - invest it in a risky venture
    - successful: return is 20%
    - failure: return is 0%
  - venture is successful if and only if total investment is at least $200

- There is one other potential investor in the venture (Beril) who is in the same situation as Ali.
- They cannot communicate and have to make the investment decision without knowing the decisions of each other.

<table>
<thead>
<tr>
<th></th>
<th>Beril Bonds</th>
<th>Beril Venture</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ali</td>
<td>110, 110</td>
<td>120, 120</td>
</tr>
<tr>
<td></td>
<td>110, 100</td>
<td>110, 110</td>
</tr>
<tr>
<td></td>
<td>100, 110</td>
<td>100, 110</td>
</tr>
</tbody>
</table>

Entrée Game

- Strategic (or Normal) Form Games
  - used if players choose their strategies without knowing the choices of others

- Extensive Form Games
  - used if some players know what others have done when playing

Investment Game with Incomplete Information

- Some players have private (and others have incomplete) information
- Ali is not certain about Beril’s preferences. He believes that she is
  - Normal with probability $p$
  - Crazy with probability $1 - p$

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<td>120, 120</td>
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<tr>
<td></td>
<td>110, 100</td>
<td>110, 110</td>
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<tr>
<td></td>
<td>100, 110</td>
<td>100, 110</td>
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<tr>
<td></td>
<td>120, 120</td>
<td>120, 120</td>
</tr>
</tbody>
</table>

The Dating Game

- Ali takes Beril out on a date
- Beril wants to marry a smart guy but does not know whether Ali is smart
- She believes that he is smart with probability 1/3
- Ali decides whether to be funny or quite
- Observing Ali’s demeanor, Beril decides what to do

Nature

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>V</td>
<td>(1/3)</td>
<td>(2/3)</td>
</tr>
<tr>
<td></td>
<td>smart</td>
<td>dumb</td>
</tr>
</tbody>
</table>

Ali

<table>
<thead>
<tr>
<th></th>
<th>marry</th>
<th>dump</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2, 1</td>
<td>-3, 1</td>
</tr>
<tr>
<td></td>
<td>0, 0</td>
<td>-1, 0</td>
</tr>
</tbody>
</table>

Beril

<table>
<thead>
<tr>
<th></th>
<th>marry</th>
<th>dump</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1/3)</td>
<td>(2/3)</td>
</tr>
<tr>
<td></td>
<td>smart</td>
<td>dumb</td>
</tr>
</tbody>
</table>

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Outline of the Course

1. Strategic Form Games
2. Dominant Strategy Equilibrium and Iterated Elimination of Dominated Actions
3. Nash Equilibrium: Theory
4. Nash Equilibrium: Applications
   4.1 Auctions
   4.2 Buyer-Seller Games
   4.3 Market Competition
   4.4 Electoral Competition
5. Mixed Strategy Equilibrium
6. Games with Incomplete Information and Bayesian Equilibrium
7. Auctions
8. Extensive Form Games: Theory
   8.1 Perfect Information Games and Backward Induction Equilibrium
   8.2 Imperfect Information Games and Subgame Perfect Equilibrium
9. Extensive Form Games: Applications
   9.1 Stackelberg Duopoly
   9.2 Bargaining
   9.3 Repeated Games
10. Extensive Form Games with Incomplete Information
    10.1 Perfect Bayesian Equilibrium
    10.2 Signaling Games
Split or Steal


- Individual players on average choose “split” 53 percent of the time
- Propensity to cooperate is surprisingly high for consequential amounts
- Less likely to cooperate if opponent has tried to vote them off previously
  - Evidence for reciprocity
- Young males are less cooperative than young females
- Old males are more cooperative than old females
Split or Steal

<table>
<thead>
<tr>
<th></th>
<th>Steal</th>
<th>Split</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Steve</strong></td>
<td>0, 0</td>
<td>100, 0</td>
</tr>
<tr>
<td><strong>Sarah</strong></td>
<td>0.100</td>
<td>50.50</td>
</tr>
</tbody>
</table>

- Set of Players $N = \{\text{Sarah, Steve}\}$
- Set of actions: $A_{\text{Sarah}} = A_{\text{Steve}} = \{\text{Steal, Split}\}$
- Payoffs

Prisoners’ Dilemma

<table>
<thead>
<tr>
<th></th>
<th>$c$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Player 1</strong></td>
<td>$-5, -5$</td>
<td>$0, -6$</td>
</tr>
<tr>
<td><strong>Player 2</strong></td>
<td>$-6, 0$</td>
<td>$-1, -1$</td>
</tr>
</tbody>
</table>

- $N = \{1, 2\}$
- $A_1 = A_2 = \{c, n\}$
- $A = \{(c, c), (c, n), (n, c), (n, n)\}$
- $u_1(c, c) = -5, u_1(c, n) = 0$, etc.

Strategic Form Games

- It is used to model situations in which players choose strategies without knowing the strategy choices of the other players
- Also known as normal form games

A strategic form game is composed of
1. Set of players: $N$
2. A set of strategies: $A_i$ for each player $i$
3. A payoff function: $u_i : A \rightarrow \mathbb{R}$ for each player $i$

$$G = (N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N})$$

- An outcome $a = (a_1, ..., a_n)$ is a collection of actions, one for each player
  - Also known as an action profile or strategy profile
- Outcome space
  $$A = \{(a_1, ..., a_n) : a_i \in A_i, i = 1, ..., n\}$$

Contribution Game

- Everybody starts with 10 TL
- You decide how much of 10 TL to contribute to joint fund
- Amount you contribute will be doubled and then divided equally among everyone
- I will distribute slips of paper that looks like this

Name:__________
Your Contribution:__________

- Write your name and an integer between 0 and 10
- We will collect them and enter into Excel
- We will choose one player randomly and pay her
  Click here for the EXCEL file
Example: Price Competition

- Toys“R”Us and Wal-Mart have to decide whether to sell a particular toy at a high or low price
- They act independently and without knowing the choice of the other store
- We can write this game in a bimatrix format

<table>
<thead>
<tr>
<th></th>
<th>Toys“R”Us</th>
<th>Wal-Mart</th>
</tr>
</thead>
<tbody>
<tr>
<td>High</td>
<td>10,10</td>
<td>2,15</td>
</tr>
<tr>
<td>Low</td>
<td>15,2</td>
<td>5,5</td>
</tr>
</tbody>
</table>

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Dominant Strategies

- $a_{-i} =$ profile of actions taken by all players other than $i$
- $A_{-i} =$ the set of all such profiles

An action $a_i$ strictly dominates $b_i$ if

$$u_i(a_i, a_{-i}) > u_i(b_i, a_{-i}) \quad \text{for all } a_{-i} \in A_{-i}$$

An action $a_i$ weakly dominates action $b_i$ if

$$u_i(a_i, a_{-i}) \geq u_i(b_i, a_{-i}) \quad \text{for all } a_{-i} \in A_{-i}$$

and

$$u_i(a_i, a_{-i}) > u_i(b_i, a_{-i}) \quad \text{for some } a_{-i} \in A_{-i}$$

An action $a_i$ is strictly dominant if it strictly dominates every action in $A_i$. It is called weakly dominant if it weakly dominates every action in $A_i$.

Dominant Strategy Equilibrium

If every player has a (strictly or weakly) dominant strategy, then the corresponding outcome is a (strictly or weakly) dominant strategy equilibrium.

<table>
<thead>
<tr>
<th></th>
<th>W</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>10,10</td>
<td>2,15</td>
</tr>
<tr>
<td>L</td>
<td>15,2</td>
<td>5,5</td>
</tr>
</tbody>
</table>

$N = \{T, W\}$

$A_T = A_W = \{H, L\}$

$u_T(H, H) = 10$

$u_W(H, L) = 15$

etc.

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Dominant Strategy Equilibrium

- A reasonable solution concept
- It only demands the players to be rational
- It does not require them to know that the others are rational too
- But it does not exist in many interesting games

Guess the Average

- We will play a game
- I will distribute slips of paper that looks like this
  
  Name: ____________
  Your guess: ____________

- Write your name and a number between 0 and 100
- We will collect them and enter into Excel
- The number that is closest to half the average wins
- Winner gets 6TL (in case of a tie we choose randomly)
  
  Click here for the EXCEL file

Beauty Contest

The Beauty Contest Thats Shaking Wall St., ROBERT J. SHILLER, NYT
3/9/2011

John Maynard Keynes supplied the answer in 1936, in “The General Theory of Employment Interest and Money,” by comparing the stock market to a beauty contest. He described a newspaper contest in which 100 photographs of faces were displayed. Readers were asked to choose the six prettiest. The winner would be the reader whose list of six came closest to the most popular of the combined lists of all readers.

The best strategy, Keynes noted, isn’t to pick the faces that are your personal favorites. It is to select those that you think others will think prettiest. Better yet, he said, move to the “third degree” and pick the faces you think that others think that still others think are prettiest. Similarly in speculative markets, he said, you win not by picking the soundest investment, but by picking the investment that others, who are playing the same game, will soon bid up higher.

New York Times online version
Price Matching

- Toys“R”Us web page has the following advertisement
  
  ![Advertisement Image]

- Sounds like a good deal for customers
- How does this change the game?

<table>
<thead>
<tr>
<th>Wal-Mart</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>High</strong></td>
</tr>
<tr>
<td>Toys“R”us</td>
</tr>
<tr>
<td><strong>Low</strong></td>
</tr>
<tr>
<td>Match</td>
</tr>
</tbody>
</table>

- Is there a dominant strategy for any of the players?
- There is no dominant strategy equilibrium for this game
- So, what can we say about this game?

- High is weakly dominated and Toys“R”us is rational
  - Toys“R”us should not use High
- High is weakly dominated and Wal-Mart is rational
  - Wal-Mart should not use High
- Each knows the other is rational
  - Toys“R”us knows that Wal-Mart will not use High
  - Wal-Mart knows that Toys“R”us will not use High
- This is where we use common knowledge of rationality

Therefore we have the following “effective” game

<table>
<thead>
<tr>
<th>Toys“R”us</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Low</strong></td>
</tr>
<tr>
<td>Match</td>
</tr>
</tbody>
</table>

- Low becomes a weakly dominated strategy for both
- Both companies will play Match and the prices will be high
- The above procedure is known as Iterated Elimination of Dominated Strategies (IEDS)

To be a good strategist try to see the world from the perspective of your rivals and understand that they will most likely do the same
Dominated Strategies

- A “rational” player should never play an action when there is another action that gives her a higher payoff irrespective of how the others play
- We call such an action a dominated action

An action $a_i$ is strictly dominated by $b_i$ if
$$u_i(a_i, a_{-i}) < u_i(b_i, a_{-i}) \quad \text{for all } a_{-i} \in A_{-i}.$$

$a_i$ is weakly dominated by $b_i$ if
$$u_i(a_i, a_{-i}) \leq u_i(b_i, a_{-i}) \quad \text{for all } a_{-i} \in A_{-i}$$
while
$$u_i(a_i, a_{-i}) < u_i(b_i, a_{-i}) \quad \text{for some } a_{-i} \in A_{-i}.$$

Iterated Elimination of Dominated Strategies

- Common knowledge of rationality justifies eliminating dominated strategies iteratively
- This procedure is known as Iterated Elimination of Dominated Strategies
- If every strategy eliminated is a strictly dominated strategy
  - Iterated Elimination of Strictly Dominated Strategies
- If IESDS leads to a unique outcome, we call the game dominance solvable
- If at least one strategy eliminated is a weakly dominated strategy
  - Iterated Elimination of Weakly Dominated Strategies

IESDS vs. IEWDS

- Order of elimination does not matter in IESDS
- It matters in IEWDS

<table>
<thead>
<tr>
<th></th>
<th>$L$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U$</td>
<td>3, 1</td>
<td>2, 0</td>
</tr>
<tr>
<td>$M$</td>
<td>4, 0</td>
<td>1, 1</td>
</tr>
<tr>
<td>$D$</td>
<td>4, 4</td>
<td>2, 4</td>
</tr>
</tbody>
</table>

- Start with $U$
- Start with $M$

Effort Game

- You choose how much effort to expend for a joint project:
  - An integer between 1 and 7
- The quality of the project depends on the smallest effort: $e$
  - Weakest link
- Effort is costly
- If you choose $e$ your payoff is
  $$6 + 2e - e$$
- We will randomly choose one round and one student and pay her
- Enter your name and effort choice
  Click here for the EXCEL file
Effort Game: 2 people 2 effort level

\[
\begin{array}{cc}
L & H \\
L & 7, 7 & 7, 1 \\
H & 1, 7 & 13, 13 \\
\end{array}
\]

- Is there a dominant strategy?
- What are the likely outcomes?
- If you expect the other to choose L, what is your best strategy (best response)?
- If you expect the other to choose H, what is your best strategy (best response)?
- \((L, L)\) is an outcome such that
  - Each player best responds, given what she believes the other will do
  - Their beliefs are correct
- It is a Nash equilibrium

Nash Equilibrium

- Nash equilibrium is a strategy profile (a collection of strategies, one for each player) such that each strategy is a best response (maximizes payoff) to all the other strategies

An outcome \(a^* = (a_1^*, ..., a_n^*)\) is a Nash equilibrium if for each player \(i\)

\[u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}) \text{ for all } a_i \in A_i\]

- Nash equilibrium is self-enforcing: no player has an incentive to deviate unilaterally
- One way to find Nash equilibrium is to first find the best response correspondence for each player
  - Best response correspondence gives the set of payoff maximizing strategies for each strategy profile of the other players
  - ... and then find where they “intersect”

Set of Nash equilibria = \{\((L, L), (H, H)\)\}

Best Response Correspondence

- The best response correspondence of player \(i\) is given by

\[B_i(a_{-i}) = \{a_i \in A_i : u_i(a_i, a_{-i}) \geq u_i(b_i, a_{-i}) \text{ for all } b_i \in A_i\}\]

- \(B_i(a_{-i})\) is a set and may not be a singleton
- In the effort game

\[
\begin{array}{cc}
L & H \\
L & 7, 7 & 7, 1 \\
H & 1, 7 & 13, 13 \\
\end{array}
\]

\[
\begin{align*}
B_1(L) &= \{L\} & B_1(H) &= \{H\} \\
B_2(L) &= \{L\} & B_2(H) &= \{H\}
\end{align*}
\]
The Bar Scene

Blonde - Brunette
Blonde 0, 0, 2, 1
Brunette 1, 2, 1, 1


Stag Hunt

Jean-Jacques Rousseau in A Discourse on Inequality

If it was a matter of hunting a deer, everyone well realized that he must remain faithful to his post; but if a hare happened to pass within reach of one of them, we cannot doubt that he would have gone off in pursuit of it without scruple...

Stag - Hare
Stag 2, 2, 0, 1
Hare 1, 0, 1, 1

Set of Nash equilibria:

\[ N(SH) = \{(S, S), (H, H)\} \]

What do you think?

How would you play this game?
Nash Demand Game

- Each of you will be randomly matched with another student
- You are trying to divide 10 TL
- Each writes independently how much she wants (in multiples of 1 TL)
- If two numbers add up to greater than 10 TL each gets nothing
- Otherwise each gets how much she wrote
- Write your name and demand on the slips
- I will match two randomly
- Choose one pair randomly and pay them

Click here for the EXCEL file

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Optimization

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) and \( D \subset \mathbb{R}^n \). A constrained optimization problem is

\[
\max f(x) \quad \text{subject to } x \in D
\]

- \( f \) is the **objective function**
- \( D \) is the **constraint set**
- A solution to this problem is \( x \in D \) such that

\[
f(x) \geq f(y) \quad \text{for all } y \in D
\]

Such an \( x \) is called a **maximizer**
- The set of maximizers is denoted

\[
\arg\max \{ f(x) | x \in D \}
\]

- Similarly for minimization problems

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A Graphical Example

![Graphical Example](image)

Example

\[
\max x^3 - 3x^2 + 2x + 1 \quad \text{subject to } 0.1 \leq x \leq 2.5
\]
Example

\[
\max -(x - 1)^2 + 2 \quad \text{s.t.} \quad x \in [0, 2].
\]

\[f(x)
\]

\[x\]

A Simple Case

Let \( f : \mathbb{R} \to \mathbb{R} \) and consider the problem \( \max_{x \in [a, b]} f(x) \).

\[f(x)
\]

\[f'(x^*) = 0
\]

\[f''(x^*) > 0
\]

We call a point \( x^* \) such that \( f'(x^*) = 0 \) a critical point.

Interior Optima

Theorem

Let \( f : \mathbb{R} \to \mathbb{R} \) and suppose \( a < x^* < b \) is a local maximum (minimum) of \( f \) on \([a, b]\). Then, \( f''(x^*) = 0 \).

- Known as first order conditions
- Only necessary for interior local optima
  - Not necessary for global optima
  - Not sufficient for local optima.
- To distinguish between interior local maximum and minimum you can use second order conditions

Recipe for solving the simple case

Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable function and consider the problem \( \max_{x \in [a, b]} f(x) \). If the problem has a solution, then it can be found by the following method:

1. Find all critical points: i.e., \( x^* \in [a, b] \) s.t. \( f'(x^*) = 0 \)
2. Evaluate \( f \) at all critical points and at boundaries \( a \) and \( b \)
3. The one that gives the highest \( f \) is the solution

- We can use Weierstrass theorem to determine if there is a solution
- Note that if \( f'(a) > 0 \) (or \( f'(b) < 0 \)), then the solution cannot be at \( a \) (or \( b \))
Example

\[
\text{max } x^2 \text{ s.t. } x \in [-1, 2].
\]

Solution

\(x^2\) is continuous and \([-1, 2]\) is closed and bounded, and hence compact. Therefore, by Weierstrass theorem the problem has a solution. \(f'(x) = 2x = 0\) is solved at \(x = 0\), which is the only critical point. We have \(f(0) = 0, f(-1) = 1, f(2) = 4\). Therefore, 2 is the global maximum.

\[
\text{Example}
\]

\[\text{max } -(x - 1)^2 + 2 \text{ s.t. } x \in [0, 2].\]

Solution

\(f\) is continuous and \([0, 2]\) is compact. Therefore, the problem has a solution. \(f'(x) = -2(x - 1) = 0\) is solved at \(x = 1\), which is the only critical point. We have \(f(1) = 2, f(0) = 1, f(2) = 1\). Therefore, 1 is the global maximum. Note that \(f'(0) > 0\) and \(f'(2) < 0\) and hence we could have eliminated 0 and 2 as candidates.

Recipe for general problems

- Generalizes to \(f: \mathbb{R}^n \to \mathbb{R}\) and the problem is
  \[
  \text{max } f(x) \text{ subject to } x \in D
  \]
- Find critical points \(x^* \in D\) such that \(Df(x^*) = 0\)
- Evaluate \(f\) at the critical points and the boundaries of \(D\)
- Choose the one that give the highest \(f\)
- Important to remember that solution must exist for this method to work
- In more complicated problems evaluating \(f\) at the boundaries could be difficult
- For such cases we have the method of the Lagrangean (for equality constraints) and Kuhn-Tucker conditions (for inequality constraints)

Cournot Duopoly

- Two firms competing by choosing how much to produce
- Augustine Cournot (1838)

Inverse demand function

\[
p(q_1 + q_2) = \begin{cases} 
  a - b(q_1 + q_2), & q_1 + q_2 \leq a/b \\
  0, & q_1 + q_2 > a/b 
\end{cases}
\]

Cost function of firm \(i = 1, 2\)

\[c_i(q_i) = cq_i\]

where \(a > c \geq 0\) and \(b > 0\)

Therefore, payoff function of firm \(i = 1, 2\) is given by

\[
u_i(q_1, q_2) = \begin{cases} 
  (a - c - b(q_1 + q_2))q_i, & q_1 + q_2 \leq a/b \\
  -cq_i, & q_1 + q_2 > a/b 
\end{cases}
\]
Claim

*Best response correspondence of firm $i \neq j$ is given by*

$$B_i(q_j) = \begin{cases} \frac{a-c-bq_j}{2b}, & q_j < \frac{a-c}{b} \\ 0, & q_j \geq \frac{a-c}{b} \end{cases}$$

**Proof.**

- If $q_2 \geq \frac{a-c}{b}$, then $u_1(q_1, q_2) < 0$ for any $q_1 > 0$. Therefore, $q_1 = 0$ is the unique payoff maximizer.
- If $q_2 < \frac{a-c}{b}$, then the best response cannot be $q_1 = 0$ (why?). Furthermore, it must be the case that $q_1 + q_2 \leq \frac{a-c}{b} \leq \frac{a}{b}$, for otherwise $u_1(q_1, q_2) < 0$. So, the following first order condition must hold

$$\frac{\partial u_1(q_1, q_2)}{\partial q_1} = a - c - 2bq_1 - bq_2 = 0$$

Similarly for firm 2.

---

**Cournot Nash Equilibrium**

In equilibrium each firm’s profit is $(a - c)^2/9b$.

Is there a way for these two firms to increase profits?

What if they form a cartel?

They will maximize

$$U(q_1 + q_2) = (a - c - b(q_1 + q_2))(q_1 + q_2)$$

Optimal level of total production is

$$q_1 + q_2 = \frac{a - c}{2b}$$

Half of the maximum total profit is

$$\frac{(a - c)^2}{8b}$$

Is the cartel stable?
Auctions

Many economic transactions are conducted through auctions

- treasury bills
- foreign exchange
- publicly owned companies
- mineral rights
- airwave spectrum rights

Also can be thought of as auctions

- takeover battles
- queues
- wars of attrition
- lobbying contests

Auction Formats

1. Open bid auctions
   1.1 ascending-bid auction
   - aka English auction
   - price is raised until only one bidder remains, who wins and pays the final price
   1.2 descending-bid auction
   - aka Dutch auction
   - price is lowered until someone accepts, who wins the object at the current price

2. Sealed bid auctions
   2.1 first price auction
   - highest bidder wins; pays her bid
   2.2 second price auction
   - aka Vickrey auction
   - highest bidder wins; pays the second highest bid
Auction Formats

Auctions also differ with respect to the valuation of the bidders

1. Private value auctions
   - each bidder knows only her own value
   - artwork, antiques, memorabilia

2. Common value auctions
   - actual value of the object is the same for everyone
   - bidders have different private information about that value
   - oil field auctions, company takeovers

Second Price Auctions

- Highest bidder wins and pays the second highest bid
- In case of a tie, the object is awarded to player 1

Strategic form:
1. \( N = \{1, 2\} \)
2. \( A_1 = A_2 = \mathbb{R}_+ \)
3. Payoff functions: For any \( (b_1, b_2) \in \mathbb{R}_+^2 \)

\[
\begin{align*}
u_1(b_1, b_2) &= \begin{cases} 
v_1 - b_2, & \text{if } b_1 \geq b_2, \\
0, & \text{otherwise}.\end{cases} \\
u_2(b_1, b_2) &= \begin{cases} v_2 - b_1, & \text{if } b_2 > b_1, \\
0, & \text{otherwise}.\end{cases}
\end{align*}
\]

Strategically Equivalent Formats

Open Bid
- English Auction
- Dutch Auction

Sealed Bid
- Second Price
- First Price

We will study sealed bid auctions
- For now we will assume that values are common knowledge
  - value of the object to player \( i \) is \( v_i \) dollars
- For simplicity we analyze the case with only two bidders
- Assume \( v_1 > v_2 > 0 \)

Second Price Auctions

I. Bidding your value weakly dominates bidding higher

Suppose your value is $10 but you bid $15. Three cases:
1. The other bid is higher than $15 (e.g. $20)
   - You loose either way: no difference
2. The other bid is lower than $10 (e.g. $5)
   - You win either way and pay $5: no difference
3. The other bid is between $10 and $15 (e.g. $12)
   - You loose with $10: zero payoff
   - You win with $15: loose $2
Second Price Auctions

II. Bidding your value weakly dominates bidding lower

Suppose your value is $10 but you bid $5. Three cases:

1. The other bid is higher than $10 (e.g. $12)
   ▶ You lose either way: no difference

2. The other bid is lower than $5 (e.g. $2)
   ▶ You win either way and pay $2: no difference

3. The other bid is between $5 and $10 (e.g. $8)
   ▶ You lose with $5: zero payoff
   ▶ You win with $10: earn $2

Weakly dominant strategy equilibrium = \((v_1, v_2)\)

There are many Nash equilibria. For example \((v_1, 0)\)

---

First Price Auctions

- Highest bidder wins and pays her own bid
- In case of a tie, the object is awarded to player 1

Strategic form:

1. \(N = \{1, 2\}\)
2. \(A_1 = A_2 = \mathbb{R}_+\)
3. Payoff functions: For any \((b_1, b_2) \in \mathbb{R}_+^2\)

\[
\begin{align*}
  u_1(b_1, b_2) &= \begin{cases} 
  v_1 - b_1, & \text{if } b_1 \geq b_2, \\
  0, & \text{otherwise}
  \end{cases} \\
  u_2(b_1, b_2) &= \begin{cases} 
  v_2 - b_2, & \text{if } b_2 > b_1, \\
  0, & \text{otherwise}
  \end{cases}
\end{align*}
\]

Necessary Conditions

Let \((b_1^*, b_2^*)\) be a Nash equilibrium. Then,
1. Player 1 wins: \(b_1^* \geq b_2^*\)

Proof

Suppose not: \(b_1^* < b_2^*\). Two possibilities:

1.1 \(b_2^* \leq v_2\): Player 1 could bid \(v_2\) and obtain a strictly higher payoff
1.2 \(b_2^* > v_2\): Player 2 has a profitable deviation: bid zero

Contradicting the hypothesis that \((b_1^*, b_2^*)\) is a Nash equilibrium.

2. \(b_1^* = b_2^*\)

Proof

Suppose not: \(b_1^* > b_2^*\). Player 1 has a profitable deviation: bid \(b_2^*\)

3. \(v_2 \leq b_2^* \leq v_1\)

Proof

Exercise
Sufficient Conditions

So, any Nash equilibrium \((b^*_1, b^*_2)\) must satisfy
\[ v_2 \leq b^*_1 = b^*_2 \leq v_1. \]

Is any pair \((b^*_1, b^*_2)\) that satisfies these inequalities an equilibrium?

Set of Nash equilibria is given by
\[ \{(b_1, b_2) : v_2 \leq b_1 = b_2 \leq v_1\} \]

Price Competition Models

- Quantity (or capacity) competition: Cournot Model
  - Augustine Cournot (1838)
- Price Competition: Bertrand Model
  - Joseph Bertrand (1883)

Two main models:
1. Bertrand Oligopoly with Homogeneous Products
2. Bertrand Oligopoly with Differentiated Products

Bertrand Duopoly with Homogeneous Products

- Two firms, each with unit cost \(c \geq 0\)
- They choose prices
  - The one with the lower price captures the entire market
  - In case of a tie they share the market equally
- Total market demand is equal to one (not price sensitive)

Strategic form of the game:
1. \(N = \{1, 2\}\)
2. \(A_1 = A_2 = \mathbb{R}_+\)
3. Payoff functions: For any \((P_1, P_2) \in \mathbb{R}_+^2\)
   \[
   u_1(P_1, P_2) = \begin{cases} 
   P_1 - c, & \text{if } P_1 < P_2, \\
   \frac{P_1 - c}{2}, & \text{if } P_1 = P_2, \\
   0, & \text{if } P_1 > P_2. 
   \end{cases} 
   \]
   \[
   u_2(P_1, P_2) = \begin{cases} 
   P_2 - c, & \text{if } P_2 < P_1, \\
   \frac{P_2 - c}{2}, & \text{if } P_2 = P_1, \\
   0, & \text{if } P_2 > P_1. 
   \end{cases} 
   \]

Nash Equilibrium

Suppose \(P^*_1, P^*_2\) is a Nash equilibrium. Then
1. \(P^*_1, P^*_2 \geq c\). Why?
2. At least one charges \(c\)
   - \(P^*_1 > P^*_2 > c\)?
   - \(P^*_2 > P^*_1 > c\)?
   - \(P^*_1 = P^*_2 > c\)?
3. \(P^*_2 > P^*_1 = c\)?
4. \(P^*_1 > P^*_2 = c\)?

The only candidate for equilibrium is \(P^*_1 = P^*_2 = c\), and it is indeed an equilibrium.

The unique Nash equilibrium of the Bertrand game is \((P^*_1, P^*_2) = (c, c)\)
Two firms with products that are imperfect substitutes

The demand functions are

\[ Q_1(p_1, p_2) = 10 - \alpha p_1 + p_2 \]
\[ Q_2(p_1, p_2) = 10 + p_1 - \alpha p_2 \]

Assume that \( \alpha > 1 \)

Unit costs are \( c \)

Exercise

Formulate as a strategic form game and find its Nash equilibria.

---

A Model of Election

Spatial Voting Models

- Candidates choose a policy
  - 10% tax rate vs. 25% tax rate
  - pro-EU vs anti-EU
- Only goal is to win the election
  - preferences: win \( \succ \) tie \( \succ \) lose
- Voters have ideal positions over the issue
  - one voter could have 15% as ideal tax rate, another 45%
- One-dimensional policy space: \([0, 1]\)
- Identify each voter with her ideal position \( t \in [0, 1] \)
- Voters’ preferences are single peaked
  - They vote for that candidate whose position is closest to their ideal point
- Society is a continuum and voters are distributed uniformly over \([0, 1]\)

Strategic Form of the Game

1. \( N = \{1, 2\} \)
2. \( A_1 = A_2 = [0, 1] \)
3. \[
   u_i(p_1, p_2) = \begin{cases} 
   1, & \text{if } i \text{ wins} \\
   \frac{1}{2}, & \text{if there is a tie} \\
   0, & \text{if } i \text{ loses}
   \end{cases}
\
\]

Say the two candidates choose \( 0 < p_1 < p_2 < 1 \)

Nash Equilibrium

Suppose \( p_1^*, p_2^* \) is a Nash equilibrium. Then

1. Outcome must be a tie
   - Whatever your opponent chooses you can always guarantee a tie
2. \( p_1^* \neq p_2^* \)?

3. \( p_1^* = p_2^* \neq 1/2 \)?

The only candidate for equilibrium is \( p_1^* = p_2^* = 1/2 \), which is indeed an equilibrium.

The unique Nash equilibrium of the election game is \( (p_1^*, p_2^*) = (1/2, 1/2) \)
This result generalizes to models with more general distributions. Equilibrium is for each party to choose the median position. Known as the median voter theorem.

Other Models
- Models with participation costs
- Models with more than two players
- Models with multidimensional policy space
- Models with ideological candidates
**Game Theory**

**Mixed Strategies**

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Koç University

---

**Matching Pennies**

<table>
<thead>
<tr>
<th>Player 1</th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>-1,1</td>
<td>1, -1</td>
</tr>
<tr>
<td>T</td>
<td>1, -1</td>
<td>-1,1</td>
</tr>
</tbody>
</table>

- How would you play?

**Goalie**

<table>
<thead>
<tr>
<th>Kicker</th>
<th>Left</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Left</td>
<td>-1,1</td>
<td>1, -1</td>
</tr>
<tr>
<td>Right</td>
<td>1, -1</td>
<td>-1,1</td>
</tr>
</tbody>
</table>

- No solution?
- You should try to be unpredictable
- Choose randomly

---

**Drunk Driving**

- Chief of police in Istanbul concerned about drunk driving.
- He can set up an alcohol checkpoint or not
  - a checkpoint always catches drunk drivers
  - but costs \( c \)
- You decide whether to drink wine or cola before driving.
  - Value of wine over cola is \( r \)
  - Cost of drunk driving is \( a \) to you and \( f \) to the city
    - incurred only if not caught
  - if you get caught you pay \( d \)

<table>
<thead>
<tr>
<th>Police</th>
<th>Check</th>
<th>No</th>
</tr>
</thead>
<tbody>
<tr>
<td>You</td>
<td>Wine</td>
<td>Cola</td>
</tr>
<tr>
<td></td>
<td>( r - d, -c )</td>
<td>( r - a, -f )</td>
</tr>
<tr>
<td></td>
<td>0, -c</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

- Assume: \( f > c > 0; d > r > a \geq 0 \)

---

**Drunk Driving**

Let’s work with numbers:

\[ f = 2, c = 1, d = 4, r = 2, a = 1 \]

So, the game becomes:

<table>
<thead>
<tr>
<th>Police</th>
<th>Check</th>
<th>No</th>
</tr>
</thead>
<tbody>
<tr>
<td>You</td>
<td>Wine</td>
<td>Cola</td>
</tr>
<tr>
<td></td>
<td>-2, -1</td>
<td>1, -2</td>
</tr>
<tr>
<td></td>
<td>0, -1</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

- What is the set of Nash equilibria?
A mixed strategy is a probability distribution over the set of actions. The police chooses to set up checkpoints with probability $1/3$. What should you do?

- If you drink cola you get 0
- If you drink wine you get $-2$ with prob. $1/3$ and 1 with prob. $2/3$
  ∗ What is the value of this to you?
  ∗ We assume the value is the expected payoff:

$$\frac{1}{3} \times (-2) + \frac{2}{3} \times 1 = 0$$

- You are indifferent between Wine and Cola
- You are also indifferent between drinking Wine and Cola with any probability

You drink wine with probability $1/2$. What should the police do?

- If he sets up checkpoints he gets expected payoff of $-1$
- If he does not

$$\frac{1}{2} \times (-2) + \frac{1}{2} \times 0 = -1$$

- The police is indifferent between setting up checkpoints and not, as well as any mixed strategy
- Your strategy is a best response to that of the police and conversely
- We have a Mixed Strategy Equilibrium

In a mixed strategy equilibrium every action played with positive probability must be a best response to other players’ mixed strategies.

- In particular players must be indifferent between actions played with positive probability
- Your probability of drinking wine $p$
- The police’s probability of setting up checkpoints $q$
- Your expected payoff to
  - Wine is $q \times (-2) + (1 - q) \times 1 = 1 - 3q$
  - Cola is 0
- Indifference condition

$$0 = 1 - 3q$$

implies $q = 1/3$

Mixed Strategy Equilibrium

The police’s expected payoff to

- Checkpoint is $-1$
- Not is $p \times (-2) + (1 - p) \times 0 = -2p$

Indifference condition

$$-1 = -2p$$

implies $p = 1/2$

$(p = 1/2, q = 1/3)$ is a mixed strategy equilibrium

Since there is no pure strategy equilibrium, this is also the unique Nash equilibrium.
Mixed and Pure Strategy Equilibria

- How do you find the set of all (pure and mixed) Nash equilibria?
- In 2 × 2 games we can use the best response correspondences in terms of the mixed strategies and plot them
- Consider the Battle of the Sexes game

\[
\begin{array}{c|cc}
& m & o \\ \hline
m & 2, 1 & 0, 0 \\ o & 0, 0 & 1, 2 \\
\end{array}
\]

- What is Player 1’s best response?
- What is Player 2’s best response?
- Expected payoff to
  - \( m \) is \( p \)
  - \( o \) is \( 2(1 - p) \)
- If \( p > 2(1 - p) \) or \( p > 2/3 \)
  - best response is \( m \) (or equivalently \( q = 1 \))
- If \( p < 2(1 - p) \) or \( p < 2/3 \)
  - best response is \( o \) (or equivalently \( q = 0 \))
- If \( p = 2(1 - p) \) or \( p = 2/3 \)
  - she is indifferent
  - best response is any \( q \in [0, 1] \)

Player 1’s best response correspondence:
\[
B_1(q) = \begin{cases} 
\{1\}, & \text{if } q > 1/3 \\
[0, 1], & \text{if } q = 1/3 \\
\{0\}, & \text{if } q < 1/3 
\end{cases}
\]

Player 2’s best response correspondence:
\[
B_2(p) = \begin{cases} 
\{1\}, & \text{if } p > 2/3 \\
[0, 1], & \text{if } p = 2/3 \\
\{0\}, & \text{if } p < 2/3 
\end{cases}
\]
**Mixed Strategies**

Player 1 mixes over $B_1(q) = \begin{cases} 1, & \text{if } q > 1/3 \\ [0, 1], & \text{if } q = 1/3 \\ 0, & \text{if } q < 1/3 \end{cases}$

Player 2 mixes over $B_2(p) = \begin{cases} 1, & \text{if } p > 2/3 \\ [0, 1], & \text{if } p = 2/3 \\ 0, & \text{if } p < 2/3 \end{cases}$

Set of Nash equilibria

$$\{(0, 0), (1, 1), (2/3, 1/3)\}$$

**Dominated Actions and Mixed Strategies**

- An easy way to figure out dominated actions is to compare expected payoffs.
- Let player 2’s mixed strategy given by $q = \text{prob}(L)$.

**Graph**

$$B_2(p) = \begin{cases} 1, & \text{if } p > 2/3 \\ [0, 1], & \text{if } p = 2/3 \\ 0, & \text{if } p < 2/3 \end{cases}$$

**Equations**

- $u_1(T, q) = 1$
- $u_1(M, q) = 3q$
- $u_1(B, q) = 4(1 - q)$

An action is a never best response if there is no belief (on $A_{-1}$) that makes that action a best response.

- $T$ is a never best response.

- An action is a NBR iff it is strictly dominated.

**What if there are no strictly dominated actions?**

- Denote player 2’s mixed strategy by $q = \text{prob}(L)$.

$$u_1(T, q) = 2, u_1(M, q) = 3q, u_1(B, q) = 3(1 - q)$$

**Pure strategy Nash eq. ($M, L$)**

- Mixed strategy equilibria?
  - Only one player mixes? Not possible
  - Player 1 mixes over $\{T, M, B\}$? Not possible
  - Player 1 mixes over $\{M, B\}$? Not possible
  - Player 1 mixes over $\{T, B\}$? Let $p = \text{prob}(T)$
    - $q = 1/3, 1 - p = p = p = 1/2$
  - Player 1 mixes over $\{T, M\}$? Let $p = \text{prob}(T)$
    - $q = 2/3, 3(1 - p) = p = p = 3/4$
Real Life Examples?

- Ignacio Palacios-Huerta (2003): 5 years’ worth of penalty kicks
- Empirical scoring probabilities

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>58, 42</td>
<td>95, 5</td>
</tr>
<tr>
<td>R</td>
<td>93, 7</td>
<td>70, 30</td>
</tr>
</tbody>
</table>

*R is the natural side of the kicker*

- What are the equilibrium strategies?

Penalty Kick

- Kicker must be indifferent
  \[58p + 95(1 - p) = 93p + 70(1 - p) \Rightarrow p = 0.42\]
- Goal keeper must be indifferent
  \[42q + 7(1 - q) = 5q + 30(1 - q) \Rightarrow q = 0.39\]

<table>
<thead>
<tr>
<th></th>
<th>Theory</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kicker</td>
<td>39%</td>
<td>40%</td>
</tr>
<tr>
<td>Goallie</td>
<td>42%</td>
<td>42%</td>
</tr>
</tbody>
</table>

Also see Walker and Wooders (2001): Wimbledon
Game Theory
Strategic Form Games with Incomplete Information

Levent Koçkesen
Koç University

Bayesian Games

We will first look at incomplete information games where players move simultaneously
- Bayesian games

Later on we will study dynamic games of incomplete information

What is new in a Bayesian game?
- Each player has a type: summarizes a player’s private information
  - Type set for player $i$: $\Theta_i$
    - A generic type: $\theta_i$
  - Set of type profiles: $\Theta = \times_{i \in N} \Theta_i$
    - A generic type profile: $\theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$

- Each player has beliefs about others’ types
  - $p_i : \Theta_i \rightarrow \Delta (\Theta_{-i})$
  - $p_i (\theta_{-i}|\theta_i)$

- Players’ payoffs depend on types
  - $u_i : A \times \Theta \rightarrow \mathbb{R}$
  - $u_i (a|\theta)$

- Different types of same player may play different strategies
  - $a_i : \Theta_i \rightarrow A_i$
  - $\alpha_i : \Theta_i \rightarrow \Delta (A_i)$

Incomplete information can be anything about the game
- Payoff functions
- Actions available to others
- Beliefs of others; beliefs of others’ beliefs of others’...

Harsanyi showed that introducing types in payoffs is adequate
Bayesian Equilibrium

Bayesian equilibrium is a collection of strategies (one for each type of each player) such that each type best responds given her beliefs about other players’ types and their strategies.

Also known as Bayesian Nash or Bayes Nash equilibrium

Bank Runs

You (player 1) and another investor (player 2) have a deposit of $100 each in a bank.

- If the bank manager is a good investor you will each get $150 at the end of the year. If not you lose your money.
- You can try to withdraw your money now but the bank has only $100 cash.
  - If only one tries to withdraw she gets $100.
  - If both try to withdraw they each can get $50.
- You believe that the manager is good with probability $q$.
- Player 2 knows whether the manager is good or bad.
- You and player 2 simultaneously decide whether to withdraw or not.

Bank Runs

The payoffs can be summarized as follows:

<table>
<thead>
<tr>
<th></th>
<th>W</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>W</td>
<td>50,50</td>
<td>100,0</td>
</tr>
<tr>
<td>N</td>
<td>0,100</td>
<td>150,150</td>
</tr>
</tbody>
</table>

Two Possible Types of Bayesian Equilibria

1. Separating Equilibria: Each type plays a different strategy
2. Pooling Equilibria: Each type plays the same strategy

Two possibilities:

1. $q < 1/2$: Player 1 chooses W. But then player 2 of Good type must play W, which contradicts our hypothesis that he plays N.
2. $q \geq 1/2$: Player 1 chooses N. The best response of Player 2 of Good type is N, which is the same as our hypothesis.

Separating Equilibria

<table>
<thead>
<tr>
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<th>N</th>
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<tbody>
<tr>
<td>W</td>
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</tr>
<tr>
<td>N</td>
<td>0,100</td>
<td>0,0</td>
</tr>
</tbody>
</table>

Separating Equilibrium

$q < 1/2$: No separating equilibrium
$q \geq 1/2$: Player 1: N, Player 2: (Good: N, Bad: W)
Pooling Equilibria

<table>
<thead>
<tr>
<th></th>
<th>W</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Good</td>
<td>50, 50</td>
<td>100, 0</td>
</tr>
<tr>
<td>Bad</td>
<td>0, 100</td>
<td>150, 150</td>
</tr>
</tbody>
</table>

1. (Good: N, Bad: N)
   - Not possible since W is a dominant strategy for Bad

2. (Good: W, Bad: W)
   - Player 1’s expected payoffs
     - W: \( q \times 50 + (1-q) \times 50 \)
     - N: \( q \times 0 + (1-q) \times 0 \)
   - Player 1 chooses W. Player 2 of Good type’s best response is W.
   - Therefore, for any value of \( q \) the following is the unique
     - Pooling Equilibrium
     - Player 1: W, Player 2: (Good: W, Bad: W)

If \( q < 1/2 \) the only equilibrium is a bank run

Cournot Duopoly with Incomplete Information about Costs

- Two firms. They choose how much to produce \( q_i \in \mathbb{R}_+ \)
- Firm 1 has high cost: \( c_H \)
- Firm 2 has either low or high cost: \( c_L \) or \( c_H \)
- Firm 1 believes that Firm 2 has low cost with probability \( \mu \in [0,1] \)
- Payoff function of player \( i \) with cost \( c_j \)
  \[
  u_i(q_1, q_2, c_j) = (a - (q_1 + q_2)) q_i - c_j q_i
  \]
- Strategies:
  \[
  q_1 \in \mathbb{R}_+ \quad q_2 : \{c_L, c_H\} \to \mathbb{R}_+
  \]

Complete Information

- Firm 1
  \[
  \max_{q_1} (a - (q_1 + q_2)) q_1 - c_H q_1
  \]
- Best response correspondence
  \[
  BR_1(q_2) = \frac{a - q_2 - c_H}{2}
  \]
- Firm 2
  \[
  \max_{q_2} (a - (q_1 + q_2)) q_2 - c_j q_2
  \]
- Best response correspondences
  \[
  BR_2(q_1, c_L) = \frac{a - q_1 - c_L}{2}
  \]
  \[
  BR_2(q_1, c_H) = \frac{a - q_1 - c_H}{2}
  \]
Incomplete Information

- Firm 2
  \[
  \max_{q_2} (a - (q_1 + q_2)) q_2 - c_j q_2
  \]

- Best response correspondences
  \[
  BR_2(q_1, c_L) = \frac{a - q_1 - c_L}{2}
  \]
  \[
  BR_2(q_1, c_H) = \frac{a - q_1 - c_H}{2}
  \]

- Firm 1 maximizes
  \[
  \mu \{ [a - (q_1 + q_2(c_L))] q_1 - c_H q_1 \}
  \]
  \[
  + (1 - \mu) \{ [a - (q_1 + q_2(c_H))] q_1 - c_H q_1 \}
  \]

- Best response correspondence
  \[
  BR_1(q_2(c_L), q_2(c_H)) = \frac{a - [\mu q_2(c_L) + (1 - \mu) q_2(c_H)] - c_H}{2}
  \]

Bayesian Equilibrium

- \[
  q_1 = \frac{a - c_H - \mu(c_H - c_L)}{3}
  \]
- \[
  q_2(c_L) = \frac{a - c_L + (c_H - c_L) - (1 - \mu)(c_H - c_L)}{3}
  \]
- \[
  q_2(c_H) = \frac{a - c_H + \mu(c_H - c_L)}{3}
  \]

- Is information good or bad for Firm 1?
- Does Firm 2 want Firm 1 to know its costs?

Complete vs. Incomplete Information
Auctions

Many economic transactions are conducted through auctions:
- Artwork
- Auctions
- Publicly owned companies
- Minerals rights
- Spectrum rights
- Government contracts

Also can be thought of as auctions:
- Bidding
- Sealed bid auctions
- War of Attrition

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Auctions

Outlines
- Auctions: Examples
- Auctions Formats
- Auctions as a Bayesian Game
- Second Price Auctions
- First Price Auctions
- Common Value Auctions
- Auction Design

Auction Formats

Auctions also differ with respect to the valuation of the bidders:
- Private value: each bidder knows only her own value.
- Common value: all bidders have beliefs about the value.

Open Bid
- Dutch Auction
- Second Price
- First Price

Closed Bid
- English Auction
- Vickrey Auction

Independent Private Values

Each bidder knows only her own valuation.

Common Value

All bidders have beliefs about the value.

Second Price Auctions

Only one 4.5G license will be sold.

First Price Auctions

Highest bidder wins and pays her bid.

Bayesian Equilibrium of First Price Auctions

Only 2 bidders:
- You are player 1 and your value is uniformly distributed over [0, 10].
- The other bidder’s values are uniformly distributed over [1, 10].

First Price Auctions

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Bayesian Equilibrium of First Price Auctions

First price auction:
- Highest bidder wins and pays her bid
- You believe the other bidder’s value is 4.5
- You believe the other bidder uses strategy
- Stronger equivalence between Dutch and First Price auctions
- Is strictly increasing and continuous
- Payoff function
Common Value Auctions and Winner’s Curse

- Suppose everybody, including you, bids their estimate and you are the winner.
- What did you just learn?
- Your estimate must have been larger than the others.
- The true value must be smaller than your estimate.
- You overpaid.
- This is known as Winner’s Curse.
- Optimal strategies are complicated.
- Bidders bid below their true value to prevent winner’s curse.

Which One Brings More Revenue?

Revenue Equivalence Theorem

Any auction with independent private values with a common distribution yields the same expected revenue.

- The number of the bidders are the same and the bidders are risk-neutral.
- The object always goes to the buyer with the highest value.
- Bidders bid lower amounts at A$5 million intervals.
- Problem: Facilitates entry deterrence.
- Turn out is higher than you thought.

Common Value Auctions

Value of a 4.5G license

- Value of between 8 and 100
- You observe v - b
- Bidder b 1 st, 2 nd, 3 rd, ... nth
- Winner pays payoff value - price
- Click here for the Excel file

Auction Design: Failures

Australian TV Licence Auction (1993)
- Two satellite TV licences
- First price sealed bid auction
- Winner’s curse
- No reserve price
- High bidder had no intention of paying.
- They defaulted.
- Choice of reserve price could be improved.
- Winner’s curse.

Auction Design: Failures

Levent Koçkesen (Koc University)

Auction Design

- Good design depends on objective.
- Revenue
- Efficiency
- Other
- One common objective is to maximize expected revenue.
- In the case of private independent values with the same number of risk-neutral bidders, optimal revenue is

Auction Design

- Correlated values: Ascending bid auction is better.
- Risk averse bidders:
  - Second price sealed bid auction.
- Coolidge: Sealed bid auctions are better to prevent collusion.
- Entry deterrence: Sealed bid auctions are better to promote entry.
- A hybrid format, such as Dutch-Auction, could be better.

Bayesian Equilibrium of First Price Auctions

- Your expected payoff if you bid b

\( (v - b)(b/a)^n-1 \)

FOC:

\( -(b/a)^n-1 + (n-1)b/a) = 0 \)

Solving for b:

\( b = \frac{v}{n} \)

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Game Theory
Extensive Form Games

Levent Koçkesen
Koç University

Entry Game

- Kodak is contemplating entering the instant photography market and Polaroid can either fight the entry or accommodate.

Strategic form games are used to model situations in which players choose strategies without knowing the strategy choices of the other players.

- In some situations players observe other players' moves before they move.

Removing Coins:
- There are 21 coins.
- Two players move sequentially and remove 1, 2, or 3 coins.
- Winner is who removes the last coin(s).
- We will determine the first mover by a coin toss.
- Volunteers?

Extensive Form Games

- Strategic form has three ingredients:
  - set of players
  - sets of actions
  - payoff functions

- Extensive form games provide more information:
  - order of moves
  - actions available at different points in the game
  - information available throughout the game

- Easiest way to represent an extensive form game is to use a game tree.
Game Trees

What’s in a game tree?

- nodes
  - decision nodes
  - initial node
  - terminal nodes
- branches
- player labels
- action labels
- payoffs
- information sets
  - to be seen later

Extensive Form Game Strategies

A pure strategy of a player specifies an action choice at each decision node of that player

Kodak’s strategies
- $S_K = \{\text{Out}, \text{In}\}$

Polaroid’s strategies
- $S_P = \{F, A\}$

Extensive Form Game Strategies

What should Polaroid do if Kodak enters?
Given what it knows about Polaroid’s response to entry, what should Kodak do?
This is an example of a backward induction equilibrium

At a backward induction equilibrium each player plays optimally at every decision node in the game tree (i.e., plays a sequentially rational strategy)

$(\text{In}, A)$ is the unique backward induction equilibrium of the entry game

- $S_1 = \{SS, SC, CS, CC\}$
- $S_2 = \{S, C\}$

Backward Induction Equilibrium

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Backward Induction Equilibrium

- What should Player 1 do if the game reaches the last decision node?
- Given that, what should Player 2 do if the game reaches his decision node?
- Given all that what should Player 1 do at the beginning?
- Unique backward induction equilibrium (BIE) is \((SS, S)\)
- Unique backward induction outcome (BIO) is \((S)\)

Power of Commitment

- Remember that \((In, A)\) is the unique backward induction equilibrium of the entry game. Polaroid’s payoff is 10.
- Suppose Polaroid commits to fight \((F)\) if entry occurs.
- What would Kodak do?
- Outcome would be \(Out\) and Polaroid would be better off
- Is this commitment credible?

Dr. Strangelove

- A U.S. air force base commander orders thirty four B-52’s to launch a nuclear attack on Soviet Union.
- He shuts off all communications with the planes and with the base.
- U.S. president invites the Russian ambassador to the war room and explains the situation.
- They decide to call the Russian president Dimitri.

Thomas Schelling

The power to constrain an adversary depends upon the power to bind oneself.
Credible Commitments: Burning Bridges

- In non-strategic environments having more options is never worse
- Not so in strategic environments
- You can change your opponent’s actions by removing some of your options
- 1066: William the Conqueror ordered his soldiers to burn their ships after landing to prevent his men from retreating
- 1519: Hernn Corts sank his ships after landing in Mexico for the same reason

Sun-tzu in *The Art of War*, 400 BC

At the critical moment, the leader of an army acts like one who has climbed up a height, and then kicks away the ladder behind him.

Strategic Form of an Extensive Form Game

- If you want to apply a strategic form solution concept
  - Nash equilibrium
  - Dominant strategy equilibrium
  - IEDS
- Analyze the strategic form of the game

Strategic form of an extensive form game

1. Set of players: $N$
   and for each player $i$
2. The set of strategies: $S_i$
3. The payoff function:
   \[ u_i : S \rightarrow \mathbb{R} \]
   where $S = \times_{i \in N} S_i$ is the set of all strategy profiles.

Extensive Form Games with Imperfect Information

- We have seen extensive form games with perfect information
  - Every player observes the previous moves made by all the players
- What happens if some of the previous moves are not observed?
  - We cannot apply backward induction algorithm anymore
  - Consider the following game between Kodak and Polaroid

Kodak doesn’t know whether Polaroid will fight or accommodate
The dotted line is an information set:
- a collection of decision nodes that cannot be distinguished by the player
- We cannot determine the optimal action for Kodak at that information set
- Set of Nash equilibria = \{(In, A), (Out, F)\}
- (Out, F) is sustained by an incredible threat by Polaroid
- Backward induction equilibrium eliminates equilibria based upon incredible threats
- Nash equilibrium requires rationality
- Backward induction requires sequential rationality
  - Players must play optimally at every point in the game
Subgame Perfect Equilibrium

We will introduce another solution concept: Subgame Perfect Equilibrium

**Definition**

A *subgame* is a part of the game tree such that

1. it starts at a single decision node,
2. it contains every successor to this node,
3. if it contains a node in an information set, then it contains all the nodes in that information set.

This is a subgame

This is not a subgame

Subgame Perfect Equilibrium

Consider the following game

The "smallest" subgame

Its strategic form

- Nash equilibrium of the subgame is (A,A)
- Reduced subgame is

- Its unique Nash equilibrium is (In)
- Therefore the unique SPE of the game is ((In,A),A)
A Simple Game

You have 10 TL to share
• A makes an offer
  ▷ $x$ for me and $10 - x$ for you
• If B accepts
  ▷ A’s offer is implemented
• If B rejects
  ▷ Both get zero
• Half the class will play A (proposer) and half B (responder)
  ▷ Proposers should write how much they offer to give responders
  ▷ I will distribute them randomly to responders
    ✴ They should write Yes or No

Click here for the EXCEL file
Bargaining

- Bargaining outcomes depend on many factors
  - Social, historical, political, psychological, etc.
- Early economists thought the outcome to be indeterminate
- John Nash introduced a brilliant alternative approach
  - **Axiomatic approach**: A solution to a bargaining problem must satisfy certain “reasonable” conditions
    - These are the axioms
    - How would such a solution look like?
  - This approach is also known as cooperative game theory
- Later non-cooperative game theory helped us identify critical strategic considerations

Let's simplify the problem

- \( u_A(x) = x \), and \( u_B(x) = x \)
- \( d_A = d_B = 0 \)
- A and B are the same in every other respect
- What is the most likely outcome?

Bargaining

- Two individuals, A and B, are trying to share a cake of size 1
- If A gets \( x \) and B gets \( y \), utilities are \( u_A(x) \) and \( u_B(y) \)
- If they do not agree, A gets utility \( d_A \) and B gets \( d_B \)
- What is the most likely outcome?

Let \( x \) be A’s share. Then

\[
\text{Slope} = 1 = \frac{1 - x - 0.4}{x - 0.3}
\]

or \( x = 0.45 \)

- So A gets 0.45 and B gets 0.55
Bargaining

In general A gets
\[ d_A + \frac{1}{2}(1 - d_A - d_B) \]
B gets
\[ d_B + \frac{1}{2}(1 - d_A - d_B) \]

But why is this reasonable?
Two answers:
1. Axiomatic: Nash Bargaining Solution
2. Non-cooperative: Alternating offers bargaining game

Nash Bargaining Solution

What if parties have different bargaining powers?
Remove symmetry axiom
Then A gets
\[ x_A = d_A + \alpha(1 - d_A - d_B) \]
B gets
\[ x_B = d_B + \beta(1 - d_A - d_B) \]
\[ \alpha, \beta > 0 \text{ and } \alpha + \beta = 1 \]
represent bargaining powers
If \( d_A = d_B = 0 \)
\[ x_A = \alpha \text{ and } x_B = \beta \]

Alternating Offers Bargaining

Two players, A and B, bargain over a cake of size 1
At time 0, A makes an offer \( x_A \in [0, 1] \) to B
\[ \text{If B accepts, A receives } x_A \text{ and B receives } 1 - x_A \]
\[ \text{If B rejects, then} \]
\[ \text{at time 1, B makes a counteroffer } x_B \in [0, 1] \]
\[ \text{If A accepts, B receives } x_B \text{ and A receives } 1 - x_B \]
\[ \text{If A rejects, he makes another offer at time 2} \]
This process continues indefinitely until a player accepts an offer
If agreement is reached at time \( t \) on a partition that gives player \( i \) a share \( x_i \)
\[ \text{player } i \text{'s payoff is } \delta_i x_i \]
\[ \delta_i \in (0, 1) \text{ is player } i \text{'s discount factor} \]
If players never reach an agreement, then each player's payoff is zero
Alternating Offers Bargaining

There is a unique solution
\[ x^*_A = \frac{1 - \delta_B}{1 - \delta_A \delta_B} \]
\[ x^*_B = \frac{1 - \delta_A}{1 - \delta_A \delta_B} \]

- There is at most one stationary no-delay SPE
- Still have to verify there exists such an equilibrium
- The following strategy profile is a SPE

Player A: Always offer \( x^*_A \), accept any \( x_B \) with \( 1 - x_B \geq \delta_A x^*_A \)
Player B: Always offer \( x^*_B \), accept any \( x_A \) with \( 1 - x_A \geq \delta_B x^*_B \)

Properties of the Equilibrium

Bargaining Power

Player A’s share
\[ \pi_A = x^*_A = \frac{1 - \delta_B}{1 - \delta_A \delta_B} \]

Player B’s share
\[ \pi_B = 1 - x^*_A = \frac{\delta_B (1 - \delta_A)}{1 - \delta_A \delta_B} \]

- Share of player \( i \) is increasing in \( \delta_i \) and decreasing in \( \delta_j \)
- Bargaining power comes from patience
- Example
\[ \delta_A = 0.9, \delta_B = 0.95 \Rightarrow \pi_A = 0.35, \pi_B = 0.65 \]
Properties of the Equilibrium

First mover advantage
If players are equally patient: \( \delta_A = \delta_B = \delta \)

\[
\pi_A = \frac{1}{1 + \delta} > \frac{\delta}{1 + \delta} = \pi_B
\]

First mover advantage disappears as \( \delta \to 1 \)

\[
\lim_{\delta \to 1} \pi_i = \lim_{\delta \to 1} \pi_B = \frac{1}{2}
\]

Nash Equilibrium of Cournot Duopoly

Best response correspondences:

\[
Q_1 = \frac{a - c - bQ_2}{2b}
\]

\[
Q_2 = \frac{a - c - bQ_1}{2b}
\]

Nash equilibrium:

\[
(Q_1^*, Q_2^*) = \left( \frac{a - c}{3b}, \frac{a - c}{3b} \right)
\]

In equilibrium each firm’s profit is

\[
\pi_1^* = \pi_2^* = \frac{(a - c)^2}{9b}
\]

Capacity Commitment: Stackelberg Duopoly

Remember Cournot Duopoly model?

- Two firms simultaneously choose output (or capacity) levels
- What happens if one of them moves first?
  - or can commit to a capacity level?

The resulting model is known as Stackelberg oligopoly

- After the German economist Heinrich von Stackelberg in *Marktform und Gleichgewicht* (1934)

The model is the same except that, now, Firm 1 moves first

Profit function of each firm is given by

\[
u_i(Q_1, Q_2) = (a - b(Q_1 + Q_2))Q_i - cQ_i
\]

Cournot Best Response Functions

\[
\begin{align*}
Q_1 &= \frac{a - c - bQ_2}{2b} \\
Q_2 &= \frac{a - c - bQ_1}{2b}
\end{align*}
\]
Stackelberg Model

The game has two stages:

1. Firm 1 chooses a capacity level $Q_1 \geq 0$
2. Firm 2 observes Firm 1’s choice and chooses a capacity $Q_2 \geq 0$

Backward Induction Equilibrium of Stackelberg Game

Sequential rationality of Firm 2 implies that for any $Q_1$ it must play a best response:

$$Q_2(Q_1) = \frac{a - c - bQ_1}{2b}$$

Firms 1’s problem is to choose $Q_1$ to maximize:

$$[a - b(Q_1 + Q_2(Q_1))]Q_1 - cQ_1$$

given that Firm 2 will best respond.

Therefore, Firm 1 will choose $Q_1$ to maximize

$$[a - b(Q_1 + \frac{a - c - bQ_1}{2b})]Q_1 - cQ_1$$

This is solved as

$$Q_1 = \frac{a - c}{2b}$$
Repeated Games

- Many interactions in the real world have an ongoing structure
  - Firms compete over prices or capacities repeatedly
- In such situations players consider their long-term payoffs in addition to short-term gains
- This might lead them to behave differently from how they would in one-shot interactions
- Consider the following pricing game in the DRAM chip industry

<table>
<thead>
<tr>
<th></th>
<th>High</th>
<th>Low</th>
</tr>
</thead>
<tbody>
<tr>
<td>Micron</td>
<td>2, 2</td>
<td>3, 0</td>
</tr>
<tr>
<td>Samsung</td>
<td>0, 3</td>
<td>1, 1</td>
</tr>
</tbody>
</table>

- What happens if this game is played only once?
- What do you think might happen if played repeatedly?

Dynamic Rivalry

- If a firm cuts its price today to steal business, rivals may retaliate in the future, nullifying the “benefits” of the original price cut
- In some concentrated industries prices are maintained at high levels
  - U.S. steel industry until late 1960s
  - U.S. cigarette industry until early 1990s
- In other similarly concentrated industries there is fierce price competition
  - Costa Rican cigarette industry in early 1990s
  - U.S. airline industry in 1992
- When and how can firms sustain collusion?
- They could formally collude by discussing and jointly making their pricing decisions
  - Illegal in most countries and subject to severe penalties

Implicit Collusion

- Could firms collude without explicitly fixing prices?
- There must be some reward/punishment mechanism to keep firms in line
- Repeated interaction provides the opportunity to implement such mechanisms
- For example Tit-for-Tat Pricing: mimic your rival’s last period price
- A firm that contemplates undercutting its rivals faces a trade-off
  - short-term increase in profits
  - long-term decrease in profits if rivals retaliate by lowering their prices
- Depending upon which of these forces is dominant collusion could be sustained
- What determines the sustainability of implicit collusion?
- Repeated games is a model to study these questions
Repeated Games

Players play a stage game repeatedly over time

- If there is a final period: finitely repeated game
- If there is no definite end period: infinitely repeated game
  - We could think of firms having infinite lives
  - Or players do not know when the game will end but assign some probability to the event that this period could be the last one

Today’s payoff of $1 is more valuable than tomorrow’s $1
  - This is known as discounting
  - Think of it as probability with which the game will be played next period
  - ... or as the factor to calculate the present value of next period’s payoff

Denote the discount factor by $\delta \in (0, 1)$

In PV interpretation: if interest rate is $r$

$$\delta = \frac{1}{1 + r}$$

Payoffs

If starting today a player receives an infinite sequence of payoffs $u_1, u_2, u_3, \ldots$

The payoff consequence is

$$(1 - \delta)(u_1 + \delta u_2 + \delta^2 u_3 + \delta^3 u_4 + \cdots)$$

Example: Period payoffs are all equal to $2$

$$(1 - \delta)(2 + \delta 2 + \delta^2 2 + \delta^3 2 + \cdots) = 2(1 - \delta)(1 + \delta + \delta^2 + \delta^3 + \cdots)$$

$$= 2(1 - \delta) \frac{1}{1 - \delta}$$

$$= 2$$

Repeated Game Strategies

Strategies may depend on history

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- **Tit-For-Tat**
  - Start with High
  - Play what your opponent played last period

- **Grim-Trigger** (called Grim-Trigger II in my lecture notes)
  - Start with High
  - Continue with High as long as everybody always played High
  - If anybody ever played Low in the past, play Low forever

Equilibria of Repeated Games

- There is no end period of the game
- Cannot apply backward induction type algorithm
- We use One-Shot Deviation Property to check whether a strategy profile is a subgame perfect equilibrium

One-Shot Deviation Property

A strategy profile is an SPE of a repeated game if and only if no player can gain by changing her action after any history, keeping both the strategies of the other players and the remainder of her own strategy constant

- Take an history
  - For each player check if she has a profitable one-shot deviation (OSD)
  - Do that for each possible history
  - If no player has a profitable OSD after any history you have an SPE
  - If there is at least one history after which at least one player has a profitable OSD, the strategy profile is NOT an SPE
Grim-Trigger Strategy Profile

There are two types of histories

1. Histories in which everybody always played High
2. Histories in which somebody played Low in some period

Histories in which everybody always played High

- Payoff to G-T
  \[
  (1 - \delta)(2 + \delta^2 + \delta^3 + \cdots) = 2(1 - \delta)(1 + \delta + \delta^2 + \delta^3 + \cdots) = 2
  \]
- Payoff to OSD (play Low today and go back to G-T tomorrow)
  \[
  (1 - \delta)(3 + \delta^2 + \delta^3 + \cdots) = (1 - \delta)(3 + \delta + \delta^2 + \delta^3 + \cdots) = 3(1 - \delta) + \delta
  \]

We need

\[2 \geq 3(1 - \delta) + \delta \quad \text{or} \quad \delta \geq 1/2\]

Histories in which somebody played Low in some period

- Payoff to G-T
  \[
  (1 - \delta)(1 + \delta + \delta^2 + \delta^3 + \cdots) = 1
  \]
- Payoff to OSD (play High today and go back to G-T tomorrow)
  \[
  (1 - \delta)(0 + \delta + \delta^2 + \delta^3 + \cdots) = (1 - \delta)\delta(1 + \delta + \delta^2 + \delta^3 + \cdots) = \delta
  \]

OSD is NOT profitable for any \(\delta\)

For any \(\delta \geq 1/2\) Grim-Trigger strategy profile is a SPE

Forgiving Trigger Strategy

Cooperative Phase

- Payoff to F-T = 2
- Payoff to OSD Outcome after a OSD
  \[
  (L, H), (L, L), (L, L), \ldots, (L, L), (H, H), (H, H), \ldots
  \]

Corresponding payoff

\[
(1 - \delta)[3 + \delta + \delta^2 + \ldots + \delta^k + 2\delta^{k+1} + 2\delta^{k+2} + \ldots] = 3 - 2\delta + \delta^{k+1}
\]

- No profitable one-shot deviation in the cooperative phase if and only if
  \[3 - 2\delta + \delta^{k+1} \leq 2\]
  \[
  \text{or} \quad \delta^{k+1} - 2\delta + 1 \leq 0
  \]

- It becomes easier to satisfy this as \(k\) becomes large

Forgiving Trigger

Grim-trigger strategies are very fierce: they never forgive

- Can we sustain cooperation with limited punishment
  - For example: punish for only 3 periods

Forgiving Trigger Strategy

- Cooperative phase: Start with \(H\) and play \(H\) if
  - everybody has always played \(H\)
  - or \(k\) periods have passed since somebody has played \(L\)
- Punishment phase: Play \(L\) for \(k\) periods if
  - somebody played \(L\) in the cooperative phase

We have to check whether there exists a one-shot profitable deviation after any history

- or in any of the two phases
Forgiving Trigger Strategy

Punishment Phase

Suppose there are \( k' \leq k \) periods left in the punishment phase.

- **Play F-T**
  \[
  (L, L), (L, L), \ldots, (L, L), (H, H), (H, H), \ldots
  \]
  \( k' \) times

- **Play OSD**
  \[
  (H, L), (L, L), \ldots, (L, L), (H, H), (H, H), \ldots
  \]
  \( k' \) times

- **F-T is better**

Forgiving Trigger strategy profile is a SPE if and only if
\[
\delta^{k+1} - 2\delta + 1 \leq 0
\]

Imperfect Detection

If your competitor cuts prices it is more likely that your sales will be lower.

- **Adopt a threshold trigger strategy**: Determine a threshold level of sales \( s \) and punishment length \( T \).
  - Start by playing High
  - Keep playing High as long as sales of both firms are above \( s \)
  - The first time sales of either firm drops below \( s \) play Low for \( T \) periods; and then restart the strategy

- **\( p_H \)**: probability that at least one firm’s sales is lower than \( s \) even when both firms choose high prices
- **\( p_L \)**: probability that the other firm’s sales are lower than \( s \) when one firm chooses low prices
  - \( p_L > p_H \)
  - \( p_H \) and \( p_L \) depend on threshold level of sales \( s \)
    - Higher the threshold more likely the sales will fall below the threshold
    - Therefore, higher the threshold higher are \( p_H \) and \( p_L \)

<table>
<thead>
<tr>
<th>Samsung</th>
<th>Micron</th>
</tr>
</thead>
<tbody>
<tr>
<td>High</td>
<td>2,2</td>
</tr>
<tr>
<td>Low</td>
<td>-1,3</td>
</tr>
</tbody>
</table>

Denote the discounted sum of expected payoff (NPV) to threshold trigger strategy by \( v \)
\[
v = 2 + \delta \left[ (1 - p_H) v + p_H \delta^T v \right]
\]

We can solve for \( v \)
\[
v = \frac{2}{1 - \delta \left[ (1 - p_H) + p_H \delta^T \right]}
\]

- **Value decreases as**
  - Threshold increases (\( p_H \) increases)
  - Punishment length increases
- You don’t want to trigger punishment too easily or punish too harshly
Imperfect Detection

What is the payoff to cheating?

\[ 3 + \delta \left[ (1 - p_L)v + p_L \delta^T v \right] \]

Threshold grim trigger is a SPE if

\[ 2 + \delta \left[ (1 - p_H)v + p_H \delta^T v \right] \geq 3 + \delta \left[ (1 - p_L)v + p_L \delta^T v \right] \]

that is

\[ \delta v (1 - \delta^T)(p_L - p_H) > 1 \]

- It is easier to sustain collusion with harsher punishment (higher $T$) although it reduces $v$
- The effect of the threshold $s$ is ambiguous: an increase in $s$
  - decreases $v$
  - may increase $p_L - p_H$

How to Sustain Cooperation?

Main conditions

- Future is important
- It is easy to detect cheaters
- Firms are able to punish cheaters

What do you do?

1. Identify the basis for cooperation
   - Price
   - Market share
   - Product design
2. Share profits so as to guarantee participation
3. Identify punishments
   - Strong enough to deter defection
   - But weak enough to be credible
4. Determine a trigger to start punishment
5. Find a method to go back to cooperation

Lysine Cartel: 1992-1995

- This is a case of an explicit collusion - a cartel
- Archer Daniels Midland (ADM) and four other companies charged with fixing worldwide lysine (an animal feed additive) price
- Before 1980s: the Japanese duopoly, Ajinomoto and Kyowa Hakko
- Expansion mid 1970s to early 1980s to America and Europe
- In early 1980s, South Korean firm, Sewon, enters the market and expands to Asia and Europe
- Prices rose to $3 per pound ($1-$2 btw 1960 and 1980)
- In early 1991 ADM and Cheil Sugar Co turned the lysine industry into a five firm oligopoly
- Prices dropped rapidly due to ADMs aggressive entry as a result of its excess capacity

Cartel Behavior

- April 1990: A, KH and S started meetings
- June 1992: five firm oligopoly formed a trade association
- Multiparty price fixing meetings amongst the 5 corporations
- Early 1993: a brief price war broke out
- 1993: establishment of monthly reporting of each company’s sales
- Prices rose in this period from 0.68 to 0.98, fell to 0.65 and rose again to above 1$
Mark Whitacre, a rising star at ADM, blows the whistle on the company’s price-fixing tactics at the urging of his wife Ginger. In November 1992, Whitacre confesses to FBI special agent Brian Shepard that ADM executives, including Whitacre himself, had routinely met with competitors to fix the price of lysine. Whitacre secretly gathers hundreds of hours of video and audio over several years to present to the FBI. Documents here: [http://www.usdoj.gov/atr/public/speeches/4489.htm](http://www.usdoj.gov/atr/public/speeches/4489.htm) [http://www.usdoj.gov/atr/public/speeches/212266.htm](http://www.usdoj.gov/atr/public/speeches/212266.htm) Criminal investigation resulted in fines and prison sentences for executives of ADM. Foreign companies settled with the United States Department of Justice Antitrust Division. Whitacre was later charged with and pled guilty to committing a $9 million fraud that occurred during the same time period he was working for the FBI.

1. Identify the basis for cooperation
   - Price
   - Market share
2. Share profits so as to guarantee participation
   - There is an annual budget for the cartel that allocates projected demand among the five
   - Prosecutors captured a scoresheet with all the numbers
   - Those who sold more than budget buy from those who sold less than budget
3. Identify punishments
   - Retaliation threat by ADM taped in one of the meetings
   - ADM has credibility as punisher: low-cost/high-capacity
   - Price cuts: 1993 price war?

When a potential predator appears, one or more sticklebacks approach to check it out. This is dangerous but provides useful information:
- If hungry predator, escape
- Otherwise stay
Milinski (1987) found that they use Tit-for-Tat like strategy:
- Two sticklebacks swim together in short spurts toward the predator
Cooperate: Move forward
Defect: Hang back
Stickleback Fish

- Milinski also ran an ingenious experiment
- Used a mirror to simulate a cooperating or defecting stickleback
- When the mirror gave the impression of a cooperating stickleback
  - The subject stickleback moved forward
- When the mirror gave the impression of a defecting stickleback
  - The subject stickleback stayed back

Vampire Bats

- Vampire bats (Desmodus rotundus) starve after 60 hours
- They feed each other by regurgitating
- Is it kin selection or reciprocal altruism?
  - Kin selection: Costly behavior that contributes to reproductive success of relatives
  - Studied them in wild and in captivity

Medieval Trade Fairs

- In 12th and 13th century Europe long distance trade took place in fairs
- Transactions took place through transfer of goods in exchange of promissory note to be paid at the next fair
- Room for cheating
- No established commercial law or state enforcement of contracts
- Fairs were largely self-regulated through Lex mercatoria, the "merchant law"
  - Functioned as the international law of commerce
  - Disputes adjudicated by a local official or a private merchant
  - But they had very limited power to enforce judgments
- Has been very successful and under lex mercatoria, trade flourished
- How did it work?
What prevents cheating by a merchant?
Could be sanctions by other merchants
But then why do you need a legal system?
What is the role of a third party with no authority to enforce judgments?

If two merchants interact repeatedly honesty can be sustained by trigger strategy
In the case of trade fairs, this is not necessarily the case
Can modify trigger strategy
- Behave honestly iff neither party has ever cheated anybody in the past
- Requires information on the other merchant’s past
- There lies the role of the third party

Milgrom, North, and Weingast (1990) construct a model to show how this can work
The stage game:
1. Traders may, at a cost, query the judge, who publicly reports whether any trader has any unpaid judgments
2. Two traders play the prisoners’ dilemma game
3. If queried before, either may appeal at a cost
4. If appealed, judge awards damages to the plaintiff if he has been honest and his partner cheated
5. Defendant chooses to pay or not
6. Unpaid judgments are recorded by the judge

If the cost of querying and appeal are not too high and players are sufficiently patient the following strategy is a subgame perfect equilibrium:
1. A trader queries if he has no unpaid judgments
2. If either fails to query or if query establishes at least one has unpaid judgement play Cheat, otherwise play Honest
3. If both queried and exactly one cheated, victim appeals
4. If a valid appeal is filed, judge awards damages to victim
5. Defendant pays judgement iff he has no other unpaid judgements

This supports honest trade
An excellent illustration the role of institutions
- An institution does not need to punish bad behavior, it just needs to help people do so
Game Theory
Extensive Form Games with Incomplete Information

Levent Koçkesen
Koç University

Signaling Examples

- Used-car dealer
  - How do you signal quality of your car?
  - Issue a warranty
- An MBA degree
  - How do you signal your ability to prospective employers?
  - Get an MBA
- Entrepreneur seeking finance
  - You have a high return project. How do you get financed?
  - Retain some equity
- Stock repurchases
  - Often result in an increase in the price of the stock
  - Manager knows the financial health of the company
  - A repurchase announcement signals that the current price is low
- Limit pricing to deter entry
  - Low price signals low cost

Extensive Form Games with Incomplete Information

- We have seen extensive form games with perfect information
  - Entry game
- And strategic form games with incomplete information
  - Auctions
- Many incomplete information games are dynamic
- There is a player with private information
- **Signaling Games**: Informed player moves first
  - Warranties
  - Education
- **Screening Games**: Uninformed player moves first
  - Insurance company offers contracts
  - Price discrimination

Signaling Games: Used-Car Market

- You want to buy a used-car which may be either good or bad (a lemon)
- A good car is worth $H$ and a bad one $L$ dollars
- You cannot tell a good car from a bad one but believe a proportion $q$ of cars are good
- The car you are interested in has a sticker price $p$
- The dealer knows quality but you don’t
- The bad car needs additional work that costs $c$ to make it look like good
- The dealer decides whether to put a given car on sale or not
- You decide whether to buy or not
- Assume
  \[ H > p > L \]
Signaling Games: Used-Car Market

We cannot apply backward induction
- No final decision node to start with
- We cannot apply SPE
- There is only one subgame - the game itself
- We need to develop a new solution concept

Bayes Law

Suppose a fair die is tossed once and consider the following events:
A: The number 4 turns up.
B: The number observed is an even number.
The sample space and the events are

\[ S = \{1, 2, 3, 4, 5, 6\} \]
\[ A = \{4\} \]
\[ B = \{2, 4, 6\} \]

\[ P(A) = 1/6, P(B) = 1/2 \]

Suppose we know that the outcome is an even number. What is the probability that the outcome is 4? We call this a conditional probability

\[ P(A | B) = \frac{1}{3} \]

Bayes Law

Given two events \( A \) and \( B \) such that \( P(B) \neq 0 \) we have

\[ P(A | B) = \frac{P(A \text{ and } B)}{P(B)} \]

Note that since

\[ P(A \text{ and } B) = P(B | A) P(A) \]

We have

\[ P(A | B) = \frac{P(B | A) P(A)}{P(B)} \]

\( A^c \): complement of \( A \)

\[ P(B) = P(B | A) P(A) + P(B | A^c) P(A^c) \]

Therefore,

\[ P(A | B) = \frac{P(B | A) P(A)}{P(B | A) P(A) + P(B | A^c) P(A^c)} \]

The probability \( P(A) \) is called the prior probability and \( P(A | B) \) is called the posterior probability.
Bayes Law: Example

A machine can be in two possible states: good (G) or bad (B)
- It is good 90% of the time
- The item produced by the machine is defective (D)
  - 1% of the time if it is good
  - 10% of the time if it is bad
- What is the probability that the machine is good if the item is defective?

\[ P(G) = 0.9, \ P(B) = 1 - 0.9 = 0.1, \ P(D | G) = 0.01, \ P(D | B) = 0.1 \]

Therefore, by Bayes’ Law

\[
P(G | D) = \frac{P(D | G) P(G)}{P(D | G) P(G) + P(D | B) P(B)}
\]

\[
= \frac{0.01 \times 0.9}{0.01 \times 0.9 + 0.10 \times 0.1}
= \frac{0.009}{0.019} \approx 0.47
\]

In this example the prior probability that the machine is in a good condition is 0.90, whereas the posterior probability is 0.47.

---

Strategies and Beliefs

A solution in an extensive form game of incomplete information is a collection of

1. A behavioral strategy profile
2. A belief system

We call such a collection an assessment
- A behavioral strategy specifies the play at each information set of the player
  - This could be a pure strategy or a mixed strategy
- A belief system is a probability distribution over the nodes in each information set

---

Perfect Bayesian Equilibrium

Sequential Rationality

At each information set, strategies must be optimal, given the beliefs and subsequent strategies

Weak Consistency

Beliefs are determined by Bayes Law and strategies whenever possible

The qualification “whenever possible” is there because if an information set is reached with zero probability we cannot use Bayes Law to determine beliefs at that information set.

Perfect Bayesian Equilibrium

An assessment is a PBE if it satisfies
1. Sequentially rationality
2. Weak Consistency
As in Bayesian equilibria we may look for two types of equilibria:

1. Pooling Equilibria: Good and Bad car dealers play the same strategy
2. Separating Equilibrium: Good and Bad car dealers play differently

Pooling Equilibria: Both Types Offer
If you buy a car with your prior beliefs your expected payoff is

\[ V = q \times (H - p) + (1 - q) \times (L - p) \geq 0 \]

- What does sequential rationality of seller imply?
  - You must be buying and it must be the case that \( p \geq c \)
- Under what conditions buying would be sequentially rational?
  \[ V \geq 0 \]

Pooling Equilibrium I
If \( p \geq c \) and \( V \geq 0 \) the following is a PBE

- Behavioral Strategy Profile: (Good: Offer, Bad: Offer), (You: Yes)
- Belief System: \( P(\text{good|offer}) = q \)

Pooling Equilibria: Both Types Hold
You must be saying No
- Otherwise Good car dealer would offer
- Under what conditions would you say No?
  \[ P(\text{good|offer}) \times (H - p) + (1 - P(\text{good|offer})) \times (L - p) \leq 0 \]
- What does Bayes Law say about \( P(\text{good|offer}) \)?
- Your information set is reached with zero probability
  - You cannot apply Bayes Law in this case
- So we can set \( P(\text{good|offer}) = 0 \)

Pooling Equilibrium II
The following is a PBE

- Behavioral Strategy Profile: (Good: Hold, Bad: Hold), (You: No)
- Belief System: \( P(\text{good|offer}) = 0 \)

This is complete market failure: a few bad apples (well lemons) can ruin a market
Separating Equilibria
Good: Offer and Bad: Hold

- What does Bayes Law imply about your beliefs?
  \[ P(good|offer) = 1 \]

- What does you sequential rationality imply?
  ▶ You say Yes
  ▶ Is Good car dealer’s sequential rationality satisfied?
    ▶ Yes
  ▶ Is Bad car dealer’s sequential rationality satisfied?
    ▶ Yes if \( p \leq c \)

Separating Equilibrium I
If \( p \leq c \) the following is a PBE
- Behavioral Strategy Profile: (Good: Offer, Bad: Hold), (You: Yes)
- Belief System: \( P(good|offer) = 1 \)

Mixed Strategy Equilibrium
The following is a little involved so let’s work with numbers
\[ H = 3000, L = 0, q = 0.5, p = 2000, c = 1000 \]

- Let us interpret player You as a population of potential buyers
- Is there an equilibrium in which only a proportion \( x, 0 < x < 1 \), of them buy a used car?
- What does sequential rationality of Good car dealer imply?
  ▶ Offer
- What does sequential rationality of buyers imply?
  ▶ Bad car dealers must Offer with positive probability, say \( b \)
- Buyers must be indifferent between Yes and No
  \[ P(good|offer)(3000 - 2000) + (1 - P(good|offer))(0 - 2000) = 0 \]
  \[ P(good|offer) = 2/3 \]

Separating Equilibria
Good: Hold and Bad: Offer

- What does Bayes Law imply about your beliefs?
  \[ P(good|offer) = 0 \]

- What does you sequential rationality imply?
  ▶ You say No
  ▶ Is Good car dealer’s sequential rationality satisfied?
    ▶ Yes
  ▶ Is Bad car dealer’s sequential rationality satisfied?
    ▶ No

There is no PBE in which Good dealer Holds and Bad dealer Offers

If \( p > c \) and \( V < 0 \) only equilibrium is complete market failure: even the good cars go unsold.

Mixed Strategy Equilibrium
The following is a PBE
- Behavioral Strategy Profile: (Good: Offer, Bad: Offer with prob. \( 1/2 \)), (You: Yes with prob. \( 1/2 \))
- Belief System: \( P(good|offer) = 2/3 \)
What is an MBA Worth?

- There are two types of workers
  - high ability (H): proportion $q$
  - low ability (L): proportion $1 - q$
- Output is equal to
  - $H$ if high ability
  - $L$ if low ability
- Workers can choose to have an MBA (M) or just a college degree (C)
- College degree does not cost anything but MBA costs
  - $c_H$ if high ability
  - $c_L$ if low ability
- Assume $c_L > H - L > c_H$
- There are many employers bidding for workers
  - Wage of a worker is equal to her expected output
- MBA is completely useless in terms of worker’s productivity!

Separating Equilibria

Only High ability gets an MBA
- What does Bayes Law imply?
  $$p_M = 1, p_C = 0$$
- What are the wages?
  $$w_M = H, w_C = L$$
- What does High ability worker’s sequential rationality imply?
  $$H - c_H \geq L$$
- What does Low ability worker’s sequential rationality imply?
  $$L \geq H - c_L$$
- Combining
  $$c_H \leq H - L \leq c_L$$
  which is satisfied by assumption
  MBA is a waste of money but High ability does it just to signal her ability

Only Low ability gets an MBA
- What does Bayes Law imply?
  $$p_M = 0, p_C = 1$$
- What are the wages?
  $$w_M = L, w_C = H$$
- What does High ability worker’s sequential rationality imply?
  $$H \geq L$$
  which is satisfied
  High ability worker is quite happy: she gets high wages and doesn’t have to waste money on MBA
- What does Low ability worker’s sequential rationality imply?
  $$L - c_L \geq H$$
  which is impossible
  Too bad for High ability workers: Low ability workers want to mimic them
  No such equilibrium: A credible signal of high ability must be costly
Pooling Equilibria

Both get an MBA

- What does Bayes Law imply?
  \[ p_M = q, \ p_C = \text{indeterminate} \]

- What are the wages?
  \[ w_M = qH + (1 - q)L, \ w_C = pCH + (1 - pC)L \]

- High ability worker’s sequential rationality imply
  \[ qH + (1 - q)L - cH \geq pCH + (1 - pC)L \]

- Low ability worker’s sequential rationality imply
  \[ qH + (1 - q)L - cL \geq pCH + (1 - pC)L \]

- Since we assumed \( cL > H - L \) the last inequality is not satisfied
- No such equilibrium: It is not worth getting an MBA for low ability workers if they cannot fool the employers.

Pooling Equilibria

Neither gets an MBA

- What does Bayes Law imply?
  \[ p_C = q, p_M = \text{indeterminate} \]

- What are the wages?
  \[ w_C = qH + (1 - q)L, w_M = pMH + (1 - pM)L \]

- High ability worker’s sequential rationality imply
  \[ qH + (1 - q)L \geq pMH + (1 - pM)L - cH \]

- Low ability worker’s sequential rationality imply
  \[ qH + (1 - q)L \geq pMH + (1 - pM)L - cL \]

- These are satisfied if \( cH \geq (p_M - q)(H - L) \). If, for example, \( p_M = q \)
- High ability workers cannot signal their ability by getting an MBA because employers do not think highly of MBAs

Signaling Recap

- Signaling works only if
  - it is costly
  - it is costlier for the bad type
- Warranties are costlier for lemons
- MBA degree is costlier for low ability applicants
- Retaining equity is costlier for an entrepreneur with a bad project
- Stock repurchases costlier for management with over-valued stock
- Low price costlier for high cost incumbent