1. (20pts.) Consider the strategic form game represented by the following bimatrix, where player 1 is the row and player 2 is the column player. Assume that \( a > 0 \).

\[
\begin{array}{ccc}
  & L & R \\
 T & a, 3 & a, 1 \\
 M & 2, 1 & 0, 0 \\
 B & 0, 0 & 1, 2 \\
\end{array}
\]

(a) (10 pts) Determine the range of values of \( a \) for which \( T \) is strictly dominated.

Solution
An action is strictly dominated if and only if it is a never best response. Let \( q \) be the probability assigned to the event that player 2 plays \( L \). Then, expected payoffs to the three actions are given by

\[
\begin{align*}
U_1(T, q) &= a \\
U_1(M, q) &= 2q \\
U_1(B, q) &= 1 - q \\
\end{align*}
\]

Note that when \( q = 1/3 \), \( U_1(M, q) = U_1(B, q) = 2/3 \) and when \( q < 1/3 \), \( U_1(B, q) > 2/3 \), whereas when \( q > 1/3 \), \( U_1(M, q) > 2/3 \) (See figure below). Therefore, if \( a < 2/3 \), \( T \) is strictly dominated.

(b) (10 pts) Assume that \( T \) is strictly dominated and find the set of all Nash equilibria (pure and mixed).

Solution
We know that a strictly dominated action is played with zero probability in any Nash equilibrium. There are two pure strategy equilibria \((M, L), (B, R)\). There is no equilibrium in which only one of the players completely mix. The only remaining possibility is both of them completely mixing. Let \( p \) be player 1’s probability of playing \( M \) and \( q \) be player 2’s probability of playing \( L \). As we know, an action receives positive probability in an equilibrium if and only if it is a best response to other players’ strategies. Therefore, given \( q \) player 1 must be indifferent between \( M \) and \( B \), and given \( p \) player 2 must be indifferent between \( L \) and \( R \). In other words,

\[
\begin{align*}
2q &= 1 - q \\
p &= 2(1 - p) \\
\end{align*}
\]
which imply \( q = 1/3 \) and \( p = 2/3 \).

Alternatively, we could have calculated the best response correspondences as follows:

\[
B_1(q) = \begin{cases} 
\emptyset, & q < 1/3 \\
[0, 1], & q = 1/3 \\
\{1\}, & q > 1/3 
\end{cases}
\]

\[
B_2(p) = \begin{cases} 
\emptyset, & p < 2/3 \\
[0, 1], & p = 2/3 \\
\{1\}, & p > 2/3 
\end{cases}
\]

and determine the set of Nash equilibria as (see figure below)

\[
(p, q) \in \{(0, 0), (1, 1), (2/3, 1/3)\}
\]

2. **(30pts.)** Two animals, player 1 and player 2, simultaneously decide whether to be hawkish (H) or dovish (D) when they are trying to share a prey. If they get into a fight they both suffer injuries, whose cost for player \( i \) is \( c_i \), \( i = 1, 2 \). There is a fight if and only if both of them choose to be hawkish. If both are hawkish or dovish they share the prey, whereas if only one is hawkish, he gets the entire prey. The value of the prey for each of them is \( v > 0 \). This game, therefore, can be represented by the following bimatrix.

\[
\begin{array}{c|cc}
 & H & D \\
\hline
H & v/2 - c_1, v/2 - c_2 & v, 0 \\
D & 0, v & v/2, v/2 \\
\end{array}
\]

It is common knowledge that \( c_1 > v/2 \), but \( c_2 \) is either 0 (player 2 is Tough) or strictly greater than \( v/2 \) (Player 2 is Weak). Player 2 knows \( c_2 \), whereas player 1 believes that \( c_2 > v/2 \) with probability \( q \in (0, 1) \) and \( c_2 = 0 \) with probability \( 1 - q \). All of this is common knowledge.

(a) **(10 pts)** Find the set of pooling pure strategy Bayesian equilibria.

**Solution**

Denote player 1’s strategy by \( a_1 \) and player 2’s strategy by \( (a_2(T), a_2(W)) \). Note that for the Tough player 2, \( H \) is a strictly dominant strategy. Therefore, in any Bayesian equilibrium he plays \( H \). The only candidate for a pooling strategy equilibrium is therefore \( a_2(T) = a_2(W) = H \). But then the best response of player 1 is \( D \), to which best responses of both types of player 2 is indeed \( H \). Therefore,

\[
a_1 = D, a_2(T) = a_2(W) = H
\]

is a pooling Bayesian equilibrium.
(b) (10 pts) Find the set of separating pure strategy Bayesian equilibria.

Solution
The only candidate is \( a_2(T) = H, a_2(W) = D \). Player 1’s expected payoff to \( H \) is given by
\[
(1 - q)(\frac{v}{2} - c_1) + qv
\]
whereas his expected payoff to \( D \) is
\[
(1 - q) \times 0 + q\frac{v}{2}
\]
Playing \( D \) is Weak player 2’s best response only if player 1 plays \( H \). This implies
\[
(1 - q)(\frac{v}{2} - c_1) + qv \geq q\frac{v}{2}
\]
which, in turn, implies that
\[
q \geq 1 - \frac{v}{2c_1}
\]
Therefore, if \( q \geq 1 - \frac{v}{2c_1} \), the following is a pure strategy separating Bayesian equilibrium
\[
a_1 = H, a_2(T) = H, a_2(W) = D
\]
whereas if \( q < 1 - \frac{v}{2c_1} \), there is no pure strategy separating Bayesian equilibrium.

(c) (10 pts) Is there an equilibrium in which player 2 chooses both \( H \) and \( D \) with positive probability?

Solution
Only the Weak type could be completely mixing. So, let \( \sigma_2(H|W) \in (0,1) \). This implies that expected payoff to \( H \) and \( D \) must be equal:
\[
\sigma_1(H)(\frac{v}{2} - c_2) + (1 - \sigma_1(H))v = (1 - \sigma_1(H))\frac{v}{2}
\]
which is solved as
\[
\sigma_1(H) = \frac{v}{2c_2} \in (0,1)
\]
Therefore, player 1 must be indifferent between \( H \) and \( D \):
\[
(1 - q)(\frac{v}{2} - c_1) + q(\frac{v}{2} - c_1) + (1 - \sigma_2(H|W))v = q(1 - \sigma_2(H|W))\frac{v}{2}
\]
which is solved as
\[
\sigma_2(H|W) = \frac{1}{q}\left(\frac{v}{2c_1} - (1 - q)\right)
\]
This is in \((0,1)\) if and only if
\[
q > 1 - \frac{v}{2c_1}
\]
Therefore, there is such an equilibrium iff this condition holds.

3. (20pts.) Consider the following second-price all-pay auction. Two players are competing to obtain an object by bidding non-negative amounts in sealed envelopes. The value of the object to player \( i \) is \( v_i \), with \( v_1 > v_2 > 0 \). The highest bidder wins the object but pays the second highest bid. In the case of a tie each player wins the object with equal probability. This auction differs from the standard second-price auction in that the loser pays her own bid, rather than paying nothing. Therefore, for any bid profile \((b_1, b_2)\), the payoff function of player \( i \) is:
\[
u_i(b_1, b_2) = \begin{cases} -b_i, & \text{if } b_i < b_j \\ \frac{1}{2}v_i - b_i, & \text{if } b_i = b_j \\ v_i - b_j, & \text{if } b_i > b_j \end{cases}
\]
where \( j \neq i \). Find the set of Nash equilibria of this game.

Solution
Let \((b_1, b_2)\) be a Nash equilibrium of this game and suppose \( b_1 = b_2 = b \). In this case player \( i \) gets \( \frac{1}{2}v_i - b \).
By conceding later than the other player, she can get \( v_i - b > \frac{1}{2}v_i - b \). So, there is no Nash equilibrium where \( b_1 = b_2 \). Suppose that \( b_1 > b_2 > 0 \) or \( b_2 > b_1 > 0 \). In either case one of the players receives...
a negative payoff and can profitably deviate by conceding immediately which gives her a payoff of zero. Therefore, either $b_1 = 0$ and $b_2 > 0$ or $b_1 = 0$ and $b_1 > 0$ in all Nash equilibria. Also observe that there is no Nash equilibrium in which $0 = b_i < b_j < v_i$ as player $i$ can increase her payoff by conceding slightly later than player $j$. Therefore, in all Nash equilibria we have either $b_1 = 0$ and $b_2 \geq v_1$ or $b_2 = 0$ and $b_1 \geq v_2$.

Let’s confirm that all such strategy profiles are Nash equilibria. In the first equilibrium player 1 receives a payoff of zero. If she deviates and concedes before player 2, she gets a negative payoff. If she concedes at the same time with or after player 2 then she receives a nonpositive payoff. So, she is best responding. Player 2’s equilibrium payoff is $v_2$ and the only other action that gives her a different payoff is to concede immediately which yields a payoff of $\frac{v_1}{2}v_2$. So, she is best responding as well. Similarly for the second equilibrium.

An alternative way to find Nash equilibria would be to calculate the best response correspondences:

$$B_i(b_j) = \begin{cases} b_i > b_j, & \text{if } b_j < v_i \\ \{0\} \text{ or } b_i > b_j, & \text{if } b_j = v_i \\ \{0\} & \text{if } b_j > v_i \end{cases}$$

where $j \neq i$.

4. (30pts.) Consider the second-price all-pay auction defined in Question 3, but this time with incomplete information. Player $i \neq j$ knows $v_i$ but is uncertain about $v_j$: she believes that $v_j$ is a random variable with uniform distribution over $[0, 1]$. All of this is common knowledge. Denote the pure strategy of player $i$ of type $v$ by $\beta_i(v)$. Find a pure strategy Bayesian equilibrium of this game, where $\beta_1(v) = \beta_2(v) = \beta(v)$ for all $v$. You may assume that $\beta$ is strictly increasing and differentiable.

Mathematical Facts:

(a) Leibniz Integral Rule:

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(x, \alpha) \, dx = \frac{db(x)}{d\alpha} f(b(x), \alpha) - \frac{da(x)}{d\alpha} f(a(x), \alpha) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial \alpha} f(x, \alpha) \, dx$$

(b) Inverse Function Theorem:

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

Solution

Suppose player 2 follows the strategy $\beta$. Then, the expected payoff of player 1 of type $v_1$ who chooses $b$ is given by

$$\int_0^{\beta^{-1}(b)} (v_1 - \beta(v_2)) dv_2 - (1 - \beta^{-1}(b)) b$$

First order condition for maximizing this function is given by

$$\frac{v_1}{\beta'(\beta^{-1}(b))} - (1 - \beta^{-1}(b)) \leq 0 \quad \text{with equality if } b > 0$$

In equilibrium we must have $\beta(0) = 0$. Since $\beta$ is strictly increasing $\beta(v) > 0$ for all $v > 0$ and therefore the FOC must hold with equality for all $v_1 > 0$. Substituting $b = \beta(v_1)$ we get

$$\beta'(v_1) = \frac{v_1}{1 - v_1}$$

for all $v_1 > 0$. By the fundamental theorem of calculus and the fact that we must have $\beta(0) = 0$ the equilibrium strategy must be

$$\beta(v) = \int_0^v \frac{x}{1 - x} \, dx$$

(2)

It can be easily verified that $\beta$ is strictly increasing and $\beta(0) = 0$.

[You could also solve it as $\beta(v) = -v - \ln(1 - v)$.]}

So, we have shown that any symmetric Bayesian equilibrium with strictly increasing strategies must take the form given in (2). We now have to show that it is indeed an equilibrium. Let $U_i(y|v)$ denote the
expected payoff of player \(i\) with type \(v\) who plays \(\beta(y)\) when every other player with type \(v_j\) plays \(\beta(v_j)\).

We will first show that \(\beta(v)\) is optimal if, and only if,

\[
U_i(v|v) \geq U_i(y|v) \quad \text{for all } y \tag{3}
\]

Clearly, if \(\beta(v)\) is optimal than (3) must hold. Now suppose that (3) holds. Since \(\beta\) is strictly increasing, continuous, and \(\beta(0) = 0\), this implies that there is no profitable deviation to \(b \leq \beta(1)\). But \(b > \beta(1)\) cannot be a profitable deviation, hence there is no profitable deviation.

Now will show that \(U_i(v|v) \geq U_i(y|v)\) for all \(y\). We can calculate \(U_i(y|v)\) as follows:

\[
U_i(y|v) = \int_0^y (v - \beta(x))dx - (1 - y) \beta(y)
\]

Therefore,

\[
\frac{\partial}{\partial y} U_i(y|v) = (v - \beta(y)) - (1 - y) \beta'(y) + \beta(y)
\]

Substituting from (1) we get

\[
\frac{\partial}{\partial y} U_i(y|v) = v - y
\]

Therefore, expected payoff is strictly increasing in \(y\) for \(y < v\) and strictly decreasing in \(y\) for \(y > v\). This implies that \(U_i(v|v) \geq U_i(y|v)\) for all \(y\).