1. Prove: Let $\Theta \neq \emptyset$ be a compact set in $\mathbb{R}^k$ and let $S_i \neq \emptyset$ be a convex and compact subset of $\mathbb{R}^{m_i}$, $i = 1, \ldots, n$. Assume that $u_i : S \times \Theta \rightarrow \mathbb{R}$ is a continuous function which is quasi-concave on $S_i$ and consider the game $G(\theta) = (N, (S_i), (u_i(\cdot, \theta)))$, where $\theta \in \Theta$. Define the correspondence $\Gamma : \Theta \rightrightarrows S$ by $\Gamma(\theta) \equiv N(G(\theta))$. Then, $\Gamma$ is a nonempty-valued correspondence with a closed graph.

Solution
The correspondence $\Gamma$ is nonempty-valued from the Nash existence theorem. To show that it has closed graph take any sequence $\theta^m$ in $\Theta$ and $s^m$ in $S$ such that $\theta^m \rightarrow \theta$, $s^m \rightarrow s$, and $s^m \in \Gamma(\theta^m)$, for all $m$. We have to show that $s \in \Gamma(\theta)$. Suppose, for contradiction, that it is not true. Then, there exists $i \in N, x_i \in S_i$ such that $\alpha \equiv u_i(x_i, s_{-i}, \theta) - u_i(s_i, s_{-i}, \theta) > 0$. Since $u_i$ is continuous for all $\varepsilon > 0$ there exists an integer $M_\varepsilon$ such that 
\[ |u_i(s_i^m, s_{-i}, \theta^m) - u_i(s_i, s_{-i}, \theta)| < \varepsilon \text{ and } |u_i(x_i, s_{-i}, \theta^m) - u_i(x_i, s_{-i}, \theta)| < \varepsilon \]
for all $m \geq M_\varepsilon$. So, for any $\varepsilon > 0$ and any $m \geq M_\varepsilon$
\[ u_i(x_i, s_{-i}, \theta^m) > u_i(x_i, s_{-i}, \theta) - \varepsilon = u_i(s_i, s_{-i}, \theta) + \alpha - \varepsilon > u_i(s_i^m, s_{-i}, \theta^m) + \alpha - 2\varepsilon \]
Setting $\varepsilon = \alpha/2$ we get $u_i(x_i, s_{-i}, \theta^m) > u_i(s_i^m, s_{-i}, \theta^m)$ for all $m \geq M_{\alpha/2}$, contradicting $s^m \in \Gamma(\theta^m)$ for all $m$.

2. Prove: Let $G = (N, (S_i), (u_i))$ be a symmetric strategic form game such that

- $S_i$ is a nonempty, convex and compact subset of $\mathbb{R}^{m_i}$
- $u_i$ is continuous on $S$ and quasi-concave on $S_i$, $i = 1, \ldots, n$

Then, there exists a strategy profile $(s^*, \ldots, s^*) \in S$ such that $(s^*, \ldots, s^*) \in N(G)$.

Solution
Denote the common set of strategies by $X$. For any $s \in X$ let $s = (s, s, \ldots, s)$ be the $n$-vector composed of the same strategy for each player. Define $B_1 : X \rightrightarrows X$ as $B_1(s) = \arg\max_{s' \in X} u_1(s', s_{-1})$ for any $s \in X$. Notice that $s^* \in B_1(s^*)$ if, and only if,
\[ u_1(s^*, s^*) \geq u_1(s, s^*), \quad \text{for all } s \in X. \]
By symmetry, this implies that
\[ u_i(s^*, s^*_{-i}) \geq u_i(s, s^*_{-i}), \quad \text{for all } s \in X. \]
which implies that $s^* \in N(G)$. Therefore, we will be done if we can show that there exists an $s^* \in X$ such that $s^* \in B_1(s^*)$. But this can be done easily in a manner similar to the proof of the Nash existence theorem.

3. Consider the following War of Attrition game. Two individuals are competing to obtain a prize. The value of the prize for individual $i$ is $v_i$ and $v_1 > v_2 > 0$. Each individual chooses the time at which she intends to give up and the one who gives up last obtains the prize. If they give up at the same time each gets half the prize. Time runs continuously from zero to infinity and competing costs one unit of payoff per unit of time. Therefore, if individual $i$ concedes first, at time $t_i \in \mathbb{R}_+$, her payoff is $-t_i$. If the other player concedes first, at time $t_j \in \mathbb{R}_+$, player $i$’s payoff is $v_i - t_j$. If both players concede at the same time, player $i$’s payoff is $\frac{1}{2}v_i - t_i$, where $t_i \in \mathbb{R}_+$ is the common concession time. Formulate this situation as a strategic form game and find its Nash equilibria.

Solution
The strategic form is given by

- $N = \{1,2\}$
4. Consider a finite population consisting of \( n \) individuals, each of whom has access to a common pool resource, say a pasture or fishing ground. Let \( x_i \) \( \in \mathbb{R}_+ \) denote the extraction effort chosen by individual \( i \), while \( X = \sum x_i \) denotes the aggregate extraction effort. Total product is given by a twice differentiable and bounded function \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) with \( f(0) = 0 \) and \( f'(0) > 0 \). Let \( A(X) \equiv f(X)/X \) stand for the average return to effort for all \( X > 0 \), and set \( A(0) \equiv f'(0) \). Assume that \( A'(X) < 0 \) for all \( X > 0 \). Each member of the population receives a share of the total product that is proportional to her share of aggregate extractive effort, i.e., \( u_i(x) = x_i A(X) \).

(a) Formulate this situation as a strategic form game.

**Solution**

- \( N = \{1, 2, \ldots, n\} \)
- \( S_i = \mathbb{R}_+ \)
- \( u_i(x) = x_i A(X) \)

(b) Find the Nash equilibria when \( n = 2 \) and

\[
f(X) = \begin{cases} X - X^2, & X \leq 1 \\ 0, & X > 1 \end{cases}
\]

What is the Pareto efficient level of aggregate extraction? Is there over-extraction or under-extraction at the Nash equilibrium? Find the Nash equilibria for arbitrary \( n \).

**Solution**

If \( x_2 < 1 \) and the best response of player 1 is strictly positive the following first order condition must hold:

\[
1 - x_2 - x_2 = 0
\]

solved as \( x_1 = (1 - x_2)/2 \). This gives a higher payoff than choosing \( x_1 = 0 \). The best response to \( x_2 \geq 1 \) is any non-negative number. Therefore,

\[
B_1(x_2) = \begin{cases} \{\frac{1 - x_2}{2}\}, & x_2 < 1 \\ \mathbb{R}_+, & x_2 \geq 1 \end{cases}
\]

(continued...
and similarly for player 2. The set Nash equilibria is given by $N = (1/3, 1/3) \cup [1, \infty)^2$

Pareto efficient level can be found by solving $\max XA(X)$. If the solution $X^* \in (0, 1)$, then the following first order condition must hold: $1 - 2X^* = 0$, solved as $X^* = 1/2$. This is indeed the solution as it gives higher aggregate payoff than $X = 0$ or $X \geq 1$. Since aggregate extraction is at least 2/3 at the Nash equilibrium, there is over extraction.

Now consider the case where there are $n$ players. Suppose $X < 1$ at the Nash equilibrium. Then each player must be choosing $x_i \in (0, 1)$ and the following first order condition must hold for each $i$:

$$1 - X - x_i = 0$$

The solution is $x_i^* = 1/(1 + n)$ for all $i$, which indeed has $X^* < 1$. There are also many equilibria with $X \geq 1$.

(c) Under what conditions on $A(X)$ is there over-extraction (compared to the Pareto efficient level) in any Nash equilibrium of the game?

Solution

A sufficient condition is $A'' \leq 0$. Pareto efficient solution $X^p$ is characterized by

$$A(X^p) + X^pA'(X^p) = 0$$

This is because $f(0) = 0$ and $f'(0) > 0$ implies that there exists an $X$ such that $XA(X) > 0$.

At any Nash equilibrium $x^*$, the following must be true:

$$A(x^*) + x^*A'(x^*) = 0, \forall x^*_i > 0$$

Since $X^* = 0$ cannot be true at any equilibrium (why?) aggregate extraction is characterized by

$$nA(X^*) + X^*A'(X^*) = 0$$

Suppose, for contradiction, that there is a Nash equilibrium aggregate extraction $X^*$ such that $X^* \leq X^p$. Then we have $A(X^*) \geq A(X^p)$ and $A'(X^*) \geq A'(X^p)$ by assumption. Therefore,

$$0 = nA(X^*) + X^*A'(X^*) \geq A(X^p) + X^*A'(X^p) = (X^* - X^p)A'(X^p) \geq 0,$$

a contradiction.