1. Prove: Let $G$ be a strategic form game. $\mathbf{D}_s(G) \subseteq \mathbf{D}_w(G) \subseteq \mathbf{N}(G)$

Solution
Let $s^* \in \mathbf{D}_s(G)$. Then, for all $i \in N$ and for any $s_i \in S_i$
\[ u_i(s^*_i, s_{-i}) > u_i(s_i, s_{-i}) \quad \text{for all } s_{-i} \in S_{-i} \]
This implies that for any $s_i \in S_i$
\[ u_i(s^*_i, s_{-i}) \geq u_i(s_i, s_{-i}) \quad \text{for all } s_{-i} \in S_{-i} \]
and
\[ u_i(s^*_i, s_{-i}) > u_i(s_i, s_{-i}) \quad \text{for some } s_{-i} \in S_{-i} \]
It follows that $s^* \in \mathbf{D}_w(G)$.
If $s^* \in \mathbf{D}_w(G)$, then for all $i \in N$ and for any $s_i \in S_i$
\[ u_i(s^*_i, s_{-i}) \geq u_i(s_i, s_{-i}) \quad \text{for all } s_{-i} \in S_{-i} \]
In particular, $u_i(s^*_i, s^*_{-i}) \geq u_i(s_i, s^*_{-i})$ for any $s_i \in S_i$. Therefore, $s^* \in \mathbf{N}(G)$

2. Prove: Let $G$ be a finite strategic form game and $\sigma^* \in \mathbf{N}(G)$. Then, $\text{supp}(\sigma^*) \subseteq \mathbf{R}(G)$, where $\text{supp}(\sigma^*) = \times_{i \in N} \text{supp}(\sigma^*_i)$.

Solution
We will prove the claim by induction. Suppose you want to prove that a proposition $P(t)$ is true for all $t = 1, 2, \ldots$ [In this case we want to prove that if $\sigma^* \in \mathbf{N}(G)$, then for any $i \in N$ and any $s_i \in \text{supp}(\sigma^*_i)$, we have $s_i \in R_i(t)$ for all $t = 1, 2, \ldots$.] This can be done by proving (a) $P(1)$ is true; and (b) $P(k)$ is true implies that $P(k + 1)$ is true. Therefore, we need to prove that if $\sigma^* \in \mathbf{N}(G)$, then for any $i \in N$ and $s_i \in \text{supp}(\sigma^*_i)$

(a) $s_i \in R_i(1)$
(b) $s_i \in R_i(k)$ implies that $s_i \in R_i(k + 1)$

Let $\sigma^* \in \mathbf{N}(G)$ and take any $i \in N$ and $s_i \in \text{supp}(\sigma^*_i)$. As we have proved in class, this implies that $U_i(s_i, \sigma^*_{-i}) \geq U_i(s'_i, \sigma^*_{-i})$ for all $s'_i \in S_i$. Since $\sigma^*_{-i} \in \times_{j \neq i} \Delta(R_j(0))$, we have $s_i \in R_i(1)$.

Now, suppose $s_i \in R_i(k)$ for all $i \in N$ and $s_i \in \text{supp}(\sigma^*_i)$. This implies, by definition, that $\sigma^*_{-i} \in \times_{j \neq i} \Delta(R_j(k))$ for all $i \in N$. Since $U_i(s_i, \sigma^*_{-i}) \geq U_i(s'_i, \sigma^*_{-i})$ for all $s'_i \in S_i$, we have that $s_i \in R_i(k + 1)$ for all $i \in N$ and $s_i \in \text{supp}(\sigma^*_i)$. Therefore, $s_i \in R_i$ for all $i \in N$ and thus $\text{supp}(\sigma^*) \subseteq \mathbf{R}(G)$.

3. Prove: Let $G$ be a finite strategic form game. If IEWDS results in a unique outcome, then this outcome must be a Nash equilibrium of $G$.

Solution
Let $s^*$ be the unique outcome that survives IEWDS and suppose, for contradiction, that it is not a Nash equilibrium. Then, there exist $i \in N$ and $s'_i \in S_i$ such that
\[ u_i(s'_i, s^*_{-i}) > u_i(s^*_i, s^*_{-i}) \quad (1) \]
Since $s^*_i$ is the only strategy that survives IEWDS, $s'_i$ must have been weakly dominated by some strategy $s''_i \neq s'_i$ at some stage $k$, i.e.,
\[ u_i(s''_i, s_{-i}) \geq u_i(s'_i, s_{-i}), \quad \text{for all } s_{-i} \text{ not yet eliminated at stage } k \]
Since $s^*$ is never eliminated, this implies that
\[ u_i(s''_i, s^*_{-i}) \geq u_i(s'_i, s^*_{-i}) \]
If \( s''_i = s^*_i \) we contradict (1). Otherwise, there exists \( s''_i \notin \{ s'_i, s^*_i \} \) such that

\[
 u_i(s''_i, s^*_{-i}) \geq u_i(s''_i, s^*_{-i})
\]

If \( s''_i = s^*_i \), we are done again. Since \( S_i \) is finite, continuing in this way we ultimately reach the desired contradiction.

4. There are \( n \geq 2 \) players each of whom simultaneously chooses an integer between 1 and 99. A player wins and gets a payoff of 1 if and only if her number is among the closest to \( 2/3 \) of the average of all the other players. Otherwise, she loses and gets a payoff of 0. Formulate this situation as a strategic form game and find the set of outcomes that survive iterated elimination of weakly dominated strategies.

**Solution**

The strategic form is given by:

- \( N = \{1, 2, \ldots, n\} \)
- \( S_i = \{1, 2, \ldots, 99\} \), for all \( i \in N \)
- Payoff function of player \( i \in N \):
  \[
  u_i(s) = \begin{cases} 
  1, & \text{if } |s_i - \frac{2}{3}s| \leq |s_j - \frac{2}{3}s| \text{ for all } j \neq i \\
  0, & \text{otherwise}
  \end{cases}
  \]

We will show that, for any \( k \geq 2 \), if the set of strategies available to each player not yet eliminated is \( \{1, 2, \ldots, k\} \), then \( k - 1 \) weakly dominates \( k \). For any \( s \in \{1, 2, \ldots, k\}^n \), let \( \bar{s}_{-i} = \sum_{j \neq i} s_j / (n - 1) \) and note that for any \( s \in \{1, 2, \ldots, k\}^n \)

\[
|s_i - \frac{2}{3}s| = \frac{1}{3n} (3n - 2)s_i - 2(n - 1)\bar{s}_{-i} |
\]

We first show that

\[
\frac{1}{3n} (3n - 2)(k - 1) - 2(n - 1)\bar{s}_{-i} \leq \frac{1}{3n} (3n - 2)(k - 2(n - 1)\bar{s}_{-i}) \tag{2}
\]

for any \( s_{-i} \in \{1, 2, \ldots, k\}^{n-1} \), so that \( u_i(k - 1, s_{-i}) \geq u_i(k, s_{-i}) \) for all \( s_{-i} \in \{1, 2, \ldots, k\}^{n-1} \). Since \( (3n - 2)(k - 2)n - 1)\bar{s}_{-i} \geq 0 \), it is immediate that (2) holds if \( (3n - 2)(k - 1) - 2(n - 1)\bar{s}_{-i} \geq 0 \). If, on the other hand, \( (3n - 2)(k - 1) - 2(n - 1)\bar{s}_{-i} < 0 \), then (2) holds if

\[
-(3n - 2)(k - 1) + 2(n - 1)\bar{s}_{-i} \leq (3n - 2)(k - 2(n - 1)\bar{s}_{-i}
\]

Simple algebra shows that this is true for any \( s_{-i} \in \{1, 2, \ldots, k\}^{n-1} \) and \( k \geq 2 \).

We now show that there exists an \( s_{-i} \in \{1, 2, \ldots, k\}^{n-1} \) such that \( u_i(k - 1, s_{-i}) > u_i(k, s_{-i}) \). Let \( s_{-i} = (k - 1, k - 1, \ldots, k - 1) \). Clearly \( u_i(k - 1, s_{-i}) = 1 \). Simple algebra shows that

\[
\frac{1}{3n} (3n - 2)(k - 2(n - 1)(k - 1)) = \frac{1}{3n} (nk + 2(n - 1)) > \frac{1}{3}(k - 1) - \frac{2}{3n} = |k - 1 - \frac{2}{3n}((n - 1)(k - 1) + k)|
\]

and thus \( u_i(k, s_{-i}) = 0 \). Therefore, the unique strategy profile that survives IEWDS is \((1, 1, \ldots, 1)\).