1. Consider the independent private values model of auctions with $n$ risk neutral bidders. Bidders’ values are independently and identically distributed over $[0, 1]$ according to distribution function $F$ which has a continuous density $f > 0$. Show that the seller’s expected revenue is the same in the first and second price auction equilibria derived in class.

Solution

Expected revenue in any of the auction formats is equal to the sum of the players’ expected payments. Therefore, let us find the expected payment by a bidder in each of the auction formats first. In the second price auction the weakly dominant strategy equilibrium is $\beta_i(v) = v$ for all $i \in N$ and $v \in [0, \bar{v}]$. Therefore, if a bidder with value $v$ wins, she expects to pay the second highest bid, which is equal to the second highest value in equilibrium, conditional on that $v$ is higher than the second highest value. Let $X$ be the random variable equal to the second highest among the $n - 1$ values and $G = F^{n-1}$ its distribution function ($g = G'$). Expected payment by a bidder with value $v$ is then given by

$$m_s(v) = G(v)E[X | X < v].$$

In the first price auction, we have derived the equilibrium bid of a player with value $v$ as $\beta(v) = E[X | X < v]$. Probability that a bidder with value $v$ wins is equal to $G(v)$ which implies that her expected payment is given by

$$m_f(v) = G(v)E[X | X < v].$$

The expected revenue in either auction is equal to each other and to

$$n \times E[m_f(V)] = n \times E[m_s(V)] = n \int_{0}^{\bar{v}} \left( \int_{0}^{x} yg(y)dy \right) f(x)dx$$

$$= n \int_{0}^{\bar{v}} \left( \int_{y}^{\bar{v}} f(x)dx \right) yg(y)dy \quad \text{[by changing the order of integration]}$$

$$= \int_{0}^{\bar{v}} y(1 - F(y))g(y)dy$$

This last expression is equal to the expectation of the second highest value among the $n$ values.

2. Consider, again, the independent private values model but suppose that the bidders are risk averse rather than risk neutral. In particular, assume that if a bidder’s value is $v$ and wins at a price $p$ her payoff is $(v - p)^{1/\alpha}$, with $\alpha \geq 1$. Let $\beta_i(v)$ denote the bid of player $i$ with value $v$.

(a) Verify that $\beta_i(v) = v$, for all $i \in N$, is a weakly dominant strategy equilibrium of the second price auction.

Solution

- $v$ dominates any $x < v$
  - i. $\max_{j \neq i} b_j \geq v > x$: payoffs to both $v$ and $x$ are zero.
  - ii. $v > x \in \max_{j \neq i} b_j$: both payoffs are equal to $(v - \max_{j \neq i} b_j)^{1/\alpha}$.
  - iii. $v > \max_{j \neq i} b_j > x$: payoff to $v$ is positive whereas the payoff to $x$ is zero.
  - iv. $v > \max_{j \neq i} b_j = x$: payoff to $v$ is $(v - \max_{j \neq i} b_j)^{1/\alpha}$ whereas the payoff to $x$ is $(v - \max_{j \neq i} b_j)^{1/\alpha} / m$, for some $m \geq 2$.
- $v$ dominates any $x > v$
  - i. $\max_{j \neq i} b_j \leq v < x$: both payoffs are equal to $(v - \max_{j \neq i} b_j)^{1/\alpha}$.
  - ii. $v < \max_{j \neq i} b_j \leq x$: payoff to $v$ is zero whereas the payoff to $x$ is negative.
  - iii. $v < x < \max_{j \neq i} b_j$: both payoffs are zero.
Solution
Suppose that player \(i\) has value \(v\). The expected payoff to bidding \(b\) given that everybody plays according to \(\beta\) is given by

\[
(v - b)^{1/\alpha} F(\beta^{-1}(b))^{n-1} = (v - b)^{1/\alpha} G(\beta^{-1}(b))
\]

where \(G = F^{n-1}\) is the distribution function of the highest value among \(n - 1\) bidders. Maximizing with respect to \(b\) yields the FOC

\[
(v - b)^{1/\alpha} G'(\beta^{-1}(b)) \frac{\beta'(\beta^{-1}(b))}{\beta' G(\beta^{-1}(b))} - \frac{1}{\alpha} (v - b)^{1/\alpha - 1} G'(\beta^{-1}(b)) = 0
\]

Substituting \(b = \beta(v)\)

\[
G(v)\beta'(v) + \alpha G'(v)\beta(v) = \alpha v G'(v)
\]

Multiplying both sides by \(G(v)^{\alpha - 1}\) and integrating we obtain

\[
\beta(v) = v - \frac{1}{\alpha G'(v)} \int_0^v G^\alpha(x) dx
\]

Conversely, we can show that playing according to \(\beta\) is a best response when others are doing so. Consider a bidder with value \(v\). Let \(U(y|v)\) be the expected payoff of a bidder with value \(v\) who bids \(\beta(y)\) rather than \(\beta(v)\):

\[
(v - \beta(y))^{1/\alpha} G(y)
\]

Bidding more than the value can never be optimal, so we can limit possible deviations to \(\beta(y) \leq v\). We have

\[
\frac{\partial}{\partial y} U(y|v) = (v - \beta(y))^{1/\alpha - 1} \left( (v - \beta(y)) G'(y) - \frac{1}{\alpha} \beta'(y) G(y) \right)
\]

From the FOC we derived before \(\frac{1}{\alpha} \beta'(y) G(y) = y G'(y) - \beta(y) G'(y)\). Substituting above, we get

\[
\frac{\partial}{\partial y} U(y|v) = (v - \beta(y))^{1/\alpha - 1} (v - y) G'(y)
\]

\[
\geq 0 \quad \text{for } y < v
\]

\[
\leq 0 \quad \text{for } y > v
\]

This shows that \(U_i(y|v)\) is indeed maximized at \(y = v\), i.e., it is a best response to play according to \(\beta\) when every other player does so.

(c) Show that as bidders become more risk averse, i.e., as \(\alpha\) increases, the seller’s expected revenue increases as well. What does that tell you about the expected revenue comparison between second price auctions and first price auctions when bidders are risk averse?

Solution
Expected revenue is equal to

\[
nE[G(v)\beta(v)]
\]

where expectation is taken with respect to \(F\). Substituting for \(\beta\)

\[
nE \left[ G(v) \left( v - \frac{1}{\alpha G'(v)} \int_0^v G^\alpha(x) dx \right) \right]
\]

Since \(\frac{G(x)}{G'(v)} \leq 1\) this function is increasing in \(\alpha\). Expected revenue is the same in second price auctions irrespective of risk preferences of the bidders whereas in first price auctions it increases as bidders become more risk averse. Therefore, with risk averse bidders expected revenue is higher in first price auctions compared to second price auctions.

3. In an all-pay auction every bidder pays her own bid, even if she does not win. Therefore, the payoff function is given by

\[
u_i(b, v_i) = \begin{cases} 
\frac{b_i}{\pi_i(b_i)} - b_i, & \text{if } b_i \geq \max_{j \neq i} b_j \\
- b_i, & \text{if } b_i < \max_{j \neq i} b_j 
\end{cases}
\]
Consider the independent private values case and find a Bayesian equilibrium in which there is a strictly increasing and differentiable $\beta : [0,1] \rightarrow \mathbb{R}$ where $\beta_i(v) = \beta(v)$, for all $i \in N$.

**Solution**

Suppose player $i$ has value $v$. Let $G = F^{n-1}$ be the distribution function of the highest value among $n-1$ bidders. Expected payoff to bidding $b$ is

$$vG(\beta^{-1}(b)) - b$$

Maximizing with respect to $b$ yields the FOC

$$v\frac{G'(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))} - 1 = 0$$

Substituting $b = \beta(v)$

$$\beta'(v) = vG'(v)$$

Integrating and using $\beta(0) = 0$ we get

$$\beta(v) = \int_0^v xG'(x)dx$$

This is only the necessary condition. Let us show that $\beta$ as defined above constitutes an equilibrium. We can restrict ourselves to deviations from $\beta(v)$ that are in the range of $\beta$. Expected payoff of player $i$ with type $v$ who plays $\beta(y)$

$$U_i(y|v) = vG(y) - \beta(y)$$

We want to show

$$U_i(v|v) \geq U_i(y|v) \text{ for all } y \geq 0$$

We have

$$\frac{\partial}{\partial y} U_i(y|v) = vG'(y) - \beta'(y)$$

From the FOC before $\beta'(y) = yG'(y)$. Therefore

$$\frac{\partial}{\partial y} U_i(y|v) = (v - y)\frac{G'(y)}{\beta(y)}$$

$$> 0 \text{ for } y < v$$

$$< 0 \text{ for } y > v$$

This shows that $U_i(y|v)$ is indeed maximized at $y = v$, i.e., it is a best response to play according to $\beta$ when every other player does so.