SENSITIVITY ANALYSIS ON A DYNAMIC PRICING PROBLEM OF AN M/M/c QUEUING SYSTEM

Eren Başar Çil, Fikri Karaesmen,
E.Lerzan Örmeci

Department of Industrial Engineering, Koç University
Rumelifeneryolu 34450 Sarıyer-İSTANBUL
ecil@ku.edu.tr, fkaraesmen@ku.edu.tr, lormeci@ku.edu.tr

Abstract: The purpose of this article is to analyze the effects of changes in the system parameters on an M/M/c queuing system in which dynamic pricing is employed as the control policy. In order to obtain the desired results, we first describe the whole system as a combination of distinct operators. Then, we show the effects of changes in the system parameters on these operators. By using our findings on the operators we work on the sensitivity analysis of a specific queuing model both theoretically and numerically.

Keywords: Dynamic Pricing, M/M/c Queue, Sensitivity Analysis, Monotone Optimal Policies, Event-Based Dynamic Programming, Long-run Average Reward

1. INTRODUCTION

The purpose of this article is to analyze the effects of changes in the system parameters, such as the service rate, the arrival rate and the number of servers on an M/M/c queuing system in which dynamic pricing is employed as the control policy. The pricing control problem of a queuing system emerges from the question of whether or not to adjust the size of the queue by enforcing a toll or an entrance fee on arriving customers. The objective of the problem can be achieving either individual, in which each customer wants to maximize his own profit, or social optimality, in which maximizing the profit of whole system is the objective. The entrance fee can be either static, not depending on the state, or dynamic, depending on the state, in the pricing control problems.

The objective of the problem in this study is to maximize the profit of the whole system while using a dynamic pricing policy. This control policy is a well-established strategy in service and production industries where typical examples include airline, telecommunications, and make-to-order manufacturing systems management.

Naor (1969) is the first researcher who discusses the pricing problem by giving quantitative arguments based on an M/M/1/k queuing model. In this work, he shows the necessity of limiting the arrivals to a queuing system by a toll to achieve the social optimality. Knudsen (1972) extends Naor's study to a multiserver queuing system. In a recent research, Ziya et al. (2002) study the effect of the customer willingness to pay, the system parameters (service and arrival rates), and the waiting room capacity on the optimal static pricing policies. The major difference between these three studies and our work is that they seek optimal prices that are independent of the state.

Low (1974a) studies the optimal dynamic pricing policies of an *M/M/c/k* queuing system. As an important result, he states the monotonicity of the optimal prices. In another study, Low (1974b) extends his work for the system with unlimited waiting room capacity. Paschalidis and Tsitsiklis (2000) work on the congestion dependent pricing by considering a service provider of a communication network or some other kinds of online services. They prove the concavity of the value function and the monotonicity of the optimal prices. Another state dependent pricing paper is introduced by Chen and Frank (2001). They work on a queuing system where a monopolist can see the length of the queue and charges a fee depending on the number of customers in the system. Gayon, et. al. (2004) assess the potential benefits of the dynamic pricing in a production inventory system with Markov Modulated demand. They compare the optimal static and dynamic pricing policies where the demand environment fluctuates. In this study, they state that dynamic pricing brings significant benefits when the environment fluctuates more significantly. The most important difference between these studies and our study is that none of them contains a complete sensitivity analysis.

An extensive sensitivity analysis for a dynamic pricing problem is performed by Gans and Savin (2004). They work on the dynamic pricing problem of a multi-server loss system. Besides the fact that their system is different from our system, we additionally analyze the effects of the number of servers on the queuing system and the optimal policy.

Finally, we review Koole (1998)'s paper which generalizes certain results about the optimal policy structure in queuing systems. In this study, he considers the whole system built by separate events and shows that if all of the events satisfy some structural properties at the same time, then the whole system will also satisfy these structural properties.

We focus on the effects of the changes in the system parameters on a queuing system. Since there are some differences in the waiting room capacity and the action space between the existing models and our model, we first work on the structural properties of the model. Then, we focus on the main contribution of this study; the sensitivity analysis. In the sensitivity analysis, we concentrate on the characteristics of the optimal prices when the system parameters change. We note that a similar analysis has been recently and independently performed by Aktaran-Kalaycı and Ayhan (2005).

Our approach is based on the event-based dynamic programming technique introduced by Koole (1998). We first divide the whole system into distinct events, namely, departure, pricing and fictitious, and define operators corresponding to each event. Then, we analyze these operators to find structural results, such as monotonicity, concavity, supermodularity and submodularity. Finally, we use these results in the sensitivity analysis.

The rest of the paper is organized as follows. In Section 2, we define our model and the operators corresponding to the events. In Section 3, we find the structural properties of these operators. Then, we perform sensitivity analysis on a multi-server queuing system with pricing control in Section 4 and present a numerical study in Section 5. Finally, we conclude and mention the future works in the last section.

2. THE MODEL

We consider an M/M/c queuing system with infinite waiting room capacity and c identical servers. Customers arrive at the system according to independent Poisson process with parameter λ and they enter to the system if their reservation price (R, the maximum price that a customer wants to pay) is more than the price charged. The service times are exponentially distributed with mean $1/\mu$. Moreover, the queue owner incurs a waiting cost, h, per customer per unit time.

At each customer arrival, the decision maker has to decide on the price to charge. The price is chosen from a set of prices in $[p_{\min}, p_{\max}]$, where this set can be considered as either continuous or discrete. We assume that the reservation price, R, is a random variable distributed with a cumulative distribution function $F_R(.)$. Accordingly, the customer does not enter the system with probability of $F_R(p)$, and he enters the system with probability of $\overline{F_R}(p) = 1\text{-}F_R(p)$.

We define the state space, the number of customers in the system, as a set of all non-negative integers $S = \{x; x \in Z^+\}$. Then, we consider $u(x, \pi)$ as the expected total discounted profit of the infinite-horizon Markov Decision Process associated to the control policy π and the initial state x. If we denote the discount rate by β, total number of customers entered the system until time t by $N(t)$, and the price at time t by $P(t)$ then:

$$u(x,\pi) = E_x^{\pi}\left[\int_0^{\infty} e^{-\beta t}\left(P(t)dN(t) - hX(t)dt\right)\right] \quad (1)$$

We are interested in finding the policy π^* that maximizes $u(x, \pi^*)$. To achieve this aim, we employ uniformization to reach an equivalent system with exponentially distributed inter-event times of rate $\mu M+\lambda+\varphi+\beta=1$ (Lippman, 1975). M, which is greater than or equal to the number of servers to be considered, and φ are introduced to ensure that the system remains in the same time scale when the system parameters are changed. For example, if the arrival rate increases by ε, φ will decrease by ε which ensures that the probability of service completion remains constant.

As we will describe some structural properties (concavity, supermodularity, etc.) of the systems which operate over an infinite horizon in this study, we first prove these structural properties with the objective of maximizing the expected total β-discounted reward for a finite number of transitions, n. The finite horizon problems allow us to use the induction to prove the structural properties for all finite n. To start the induction we specify the initial function $u_0(x)$ as $u_0(x)=0$ for all states x. $u_n(x)$ is the maximum expected total β-discounted reward of the system starting in state x with n transitions remaining in the future and the optimality equation of the finite horizon problem is:

$$u_{n+1}(x) = \mu M T_D u_n(x) + \lambda T_P u_n(x) + \varphi T_F u_n(x) - hx \quad (2)$$

where,

$$T_D u(x) = \begin{cases} \dfrac{x}{M} u(x-1) + \left(1 - \dfrac{x}{M}\right) u(x) & if\ x \le c \\ \dfrac{c}{M} u(x-1) + \left(1 - \dfrac{c}{M}\right) u(x) & if\ x > c \end{cases}$$

$$T_P u(x) = \max_p \left\{ \overline{F_R}(p) \left[p + u(x+1) \right] + F_R(p) u(x) \right\}$$

$$T_F u(x) = u(x)$$

By using the standard arguments of Markov Decision theory (See (Puterman, 1994)), there exists an optimal stationary policy for the infinite horizon problem and $u(x) = \lim_{n \to \infty} u_n(x)$ whenever $\beta > 0$. $u(x)$ is the value function of the infinite horizon problem. Therefore, structural results obtained for $u_n(x)$ hold for $u(x)$. Moreover, these structural results are also true for the average reward criterion as a result of the conditions introduced by Weber and Stidham (1987). The optimality equation for the average reward criterion is as follows, where g^* is the optimal expected revenue per unit time and $u'(x)$ is the relative value function:

$$g^* + u'(x) = \mu M T_D u'(x) + \lambda T_P u'(x) + \varphi T_F u'(x) - hx \quad (3)$$

3. STRUCTURAL PROPERTIES PRESERVED BY THE OPERATORS

3.1 Monotonicity and Concavity

Monotonicity of the value function implies that the expected total discounted reward of the system with initially x customers, $u(x)$, is higher than that of the system with one more customer, $u(x+1)$. In other words, adding one more customer to the system has a positive cost. We use the term opportunity cost of a new customer to describe this cost, $u(x)-u(x+1)$. Since we use event-based dynamic programming, we focus on the monotonicity of $Tu(x)$, the result of a certain operator T, in order to show the existence of the opportunity cost of a new customer.

After the fictitious event, the system remains in the same state, therefore, if $u(x)$ is monotone, $T_F u(x)$ will obviously be monotone and the monotonicity of $T_D u(x)$ is proven by Koole (1998). To our knowledge, the monotonicity of $T_P u(x)$ for an $M/M/c$ queuing system has not been shown before. Thus, we prove this property for $T_P u(x)$. The complete proof and all of the other proofs can be seen in our technical appendix on the web[1]. The following lemma summarizes our findings about the results obtained for the monotonicity property.

Lemma 1: If $u(x)$ is non-increasing in x, then $T_F u(x)$, $T_D u(x)$, and $T_P u(x)$ will be non-increasing in x.

The optimal price for a given state x, $p^*(x)$ can be calculated by the derivative of $T_P u(x)$. As a result of this derivation, we obtain Equation 4 and $p^*(x)$ is the root of this equation;

$$p^*(x) = \left(u(x) - u(x+1) \right) + \frac{1}{r(p^*(x))} \quad (4)$$

where $r(.)$ is the hazard rate of $F_R(.)$ See (Ziya et. al., 2002) about the hazard rate.

In this equation, it is obvious that the higher the opportunity cost of a new customer, the higher the optimal price is charged. As a result of this relation, we work on the concavity of $u(x)$ which is equivalent to the monotonicity of the opportunity cost of a new customer.

As in the monotonicity property, the concavity of $T_F u(x)$ is obvious and the concavity of $T_D u(x)$ is proven by Koole (1998). We, again, need to prove the concavity of $T_P u(x)$ and this proof is in our technical appendix. The operators preserve concavity of $u(x)$ as follows;

Lemma 2: If $u(x)$ is non-increasing and concave in x, then $T_F u(x)$, $T_D u(x)$, and $T_P u(x)$ will be concave in x.

3.2 Submodularity

In this section, we observe the effects of changes in the service rate, μ, and the arrival rate, λ, on the opportunity cost of a new customer. We define $u^i(x)$ as the value function of system i in which α_i is the parameter whose effects we would like to observe. Then, we compare $u^1(x)-u^1(x+1)$ and $u^2(x)-u^2(x+1)$, which are the opportunity cost of a new customer in system 1 and system 2 respectively whenever $\alpha_1 \le \alpha_2$.

In this paper, $u(x)$ is defined to be submodular with respect to α and x, if $u^1(x)-u^1(x+1) \le u^2(x)-u^2(x+1)$, whenever $\alpha_1 \le \alpha_2$. This property implies that the opportunity cost of a new customer increases when the system parameter α, either μ or λ increases. We do not analyze the effects of the number of servers, c, in this case because the departure operator does not

[1] http://home.ku.edu.tr/~lormeci/COK-INCOM-TA-06.pdf

preserve the submodularity of $u(x)$ with respect to c and x due to a counter example.

$T_F u(x)$ is the same as $u(x)$, so it is submodular when $u(x)$ is submodular. Since the submodularity of $T_D u(x)$ and $T_P u(x)$ have not been studied for an *M/M/c* queuing system, we prove that both operators preserve the submodularity of $u(x)$. The result of the submodularity property can be summarized as the following lemma.

Lemma 3: Given that $u(x)$ is non-increasing and concave in x. If $u(x)$ is submodular with respect to α and x, then $T_F u(x)$, $T_D u(x)$, and $T_P u(x)$ will also be submodular with respect to α and x.

3.3 Supermodularity

Supermodularity is the reverse of submodularity, and implies that the opportunity cost of a new customer decreases when the parameter increases. Hence, we define $u(x)$ as supermodular with respect to α and x, if $u^1(x)–u^1(x+1)\geq u^2(x)–u^2(x+1)$whenever $\alpha_1\leq\alpha_2$.

Another difference between submodularity and supermodularity is that the departure operator can also preserve the supermodularity of $u(x)$ with respect to c and x whereas it can not preserve submodularity. Therefore, α can be c as well as μ and λ in this subsection.

As in all cases, $T_F u(x)$ preserves the supermodularity of $u(x)$ and we prove that $T_D u(x)$ and $T_P u(x)$ are also supermodular with respect to α and x if $u(x)$ is supermodular. We state the results about supermodularity as in Lemma 4.

Lemma 4: Given that $u(x)$ is non-increasing and concave in x. If $u(x)$ is supermodular with respect to α and x, then $T_F u(x)$, $T_D u(x)$, and $T_P u(x)$ will also be supermodular with respect to α and x.

3.4 Monotonicity of Tu(x) -u(x)

During the observation of the effects of a parameter, we increase this parameter by ε. Then, the probability of occurrence of the event related with this parameter increases by ε whereas the probability of occurrence of the fictitious event decreases by ε. For example, if the arrival rate increases by ε, the occurrence probability of the pricing event will increase by ε and the occurrence probability of the fictitious event decreases by ε. Thus, we obtain an additional reward of, $Tu(x)$-$u(x)$ with probability ε.

In the previous two sections, we find that the operators corresponding to the distinct events preserve both supermodular and submodular of $u(x)$ with respect to the service and the arrival rates. This result is necessary because all events operators should preserve supermodularity (submodularity) in order to prove the supermodularity (submodularity) of $u(x)$. At this point, one can ask how the effects of changes in the system parameters are different from each other. The answer is the monotonicity of $Tu(x)$-$u(x)$.

The monotonicity of $Tu(x)$-$u(x)$ differs according to the characteristics of the event. To illustrate, the additional reward obtained by the departure of one customer is more valuable when there is one more customer, so that $T_D u(x)–u(x)$ is non-decreasing in x. On the other hand, $T_P u(x)–u(x)$ is non-increasing in x because the more crowded the system is, the less additional reward we will obtain by an arrival of a new customer. Lemma 5 is the summary of the characteristics of the $Tu(x)$-$u(x)$.

Lemma 5: Given that $u(x)$ is non-increasing and concave in x, then

a) $T_D u(x)$- $u(x)$ is non-decreasing in x,
b) $T_P u(x) – u(x)$ is non-increasing in x.

4. STRUCTURAL PROPERTIES OF THE VALUE FUNCTION AND SENSITIVITY ANALYSIS

4.1 Monotonicity and Concavity of u(x)

In order to prove the monotonicity and the concavity of $u(x)$, we first prove these properties of $u_n(x)$ for all finite n by induction. We use Lemmas 1 and 2 to show the monotonicity and the concavity of $u_n(x)$ and then, extend these results for $u(x)$ by using the relationship between $u_n(x)$ and $u(x)$. Furthermore, concavity of $u(x)$ implies that optimal prices are non-decreasing in x. The following theorem summarizes the structure of the model and the optimal policy.

Theorem 1: $u(x)$ and $u'(x)$ are non-increasing and concave in x. Moreover, the optimal price for the state x, $p^*(x)$, is a non-decreasing function of x for both the discounted and average reward criteria.

4.2 Effect of Changes in the Service Rate, μ

The first parameter that we analyze is the service rate. In this analysis, we examine the effect of an increase in μ on the system. Intuitively, if the service rate increases, the system will process faster and the opportunity costs of a new customer will reduce. As we defined before, supermodularity implies this intuition. Therefore, we try to prove the supermodularity of the system with respect to μ and x.

$T_D u(x)$-$u(x)$, the additional reward obtained as a result of an increase in μ, is the most crucial part of the existence of the supermodularity of the whole system. By Lemma 5, this reward is increasing in x, and it means that we have more additional reward in state $x+1$ than we have in state x. Thus, if μ increases then being in state $x+1$ will become more desirable. In other words, the opportunity cost of a new customer decreases by an increase in μ. Since the opportunity cost of a new customer affects the optimal prices directly, the optimal prices reduce when μ increases. The following theorem briefly explains our findings on the effect of changes in μ.

Theorem 2: u(x) and *u'(x)* are supermodular with respect to μ and x. Moreover, the optimal price for the state x, $p^*(x)$, is non-increasing in μ for both the discounted and average reward criteria.

4.3 Effect of Changes in the Arrival Rate, λ

The effect of the arrival rate on the congestion of the system is the reverse of the effect of μ. Thus, we expect that the opportunity cost of a new customer is non-decreasing in λ. Because of this intuition, we work on the submodularity of the system with respect to λ and x.

The additional reward obtained when λ increases, $T_P u(x)-u(x)$, is a non-increasing function of x, so that being in state $x+1$ becomes less attractive after an increase in λ. Therefore, the opportunity cost of a new customer increases when λ increases and this implies the submodularity of the system. The following theorem summarizes the effect of λ on the system.

THEOREM.4: u(x) and *u'(x)* are submodular with respect to λ and x. Moreover, the optimal price for the state x, $p^*(x)$, is non-decreasing in λ for both the discounted and average reward criteria.

4.4 Effect of Changes in the Number of Servers, c

As both c and μ affect the congestion of the system in the same manner, the effects of changes in these parameters on the system are similar. Therefore, we work on the supermodularity of the system with respect to c and x.

THEOREM.5: *u(x)* and *u'(x)* are supermodular with respect to c and x. Moreover, the optimal price for the state x, $p^*(x)$, is non-increasing in c for both the discounted and average reward criteria.

5. NUMERICAL STUDY

We consider a job shop manufacturing system with the objective of maximizing the expected long-run average profit. The arrival process is a Poisson process with a rate of 5 customers per day, and the service completion times are exponentially distributed with a mean of 1/6 customers per day. The reservation price distribution is a uniform distribution with parameters 100 and 200. Moreover, the queue owner incurs a holding cost of \$250 per customer per day. Both M and φ are set as 10 to ensure that $\mu M+\lambda+\varphi=75$. Since the objective is maximizing average profit, we assume that $\beta \to 0$. After uniformization, the system becomes a discrete-time Markov chain in which a transition occurs every 1/75 days.

The optimal policy is numerically obtained by the value iteration procedure with a precision value, 0.0001. Since the waiting room capacity is infinite in our model, we truncate the state space at 500 to implement the value iteration. In the first part of the numerical study, we increase the number of servers from 2 to 3. Optimal prices for both cases are represented in Table 1.

Table 1The effect of changes in number of servers

# of customer in the system	Optimal prices of System 1 (with 2 servers)	Optimal prices of System 2 (with 3 servers)
0	122.64	121.07
1	125.49	121.44
2	138.78	122.94
3	151.31	131.36
4	163.20	139.55
5	174.52	147.55
6	185.35	155.36
7	195.80	163.00
8	200.00	170.47
9	200.00	177.78
10	200.00	184.95
11	200.00	191.98
12	200.00	198.93
13	200.00	200.00
14	200.00	200.00
15	200.00	200.00

Since the optimal price is an increasing function of the number of customers in the system, it is not necessary to show the optimal policy for all states in Table 1. Optimal prices for the states $x>15$ are all 200, the maximum price, for both cases.

As it can be easily seen in Table 1, the optimal prices decrease when c increases. After c increases, the system starts to operate with a lower load, and thus the opportunity cost of a new customer decreases. Therefore, the queue owner reduces the prices to encourage new customers to enter the system.

In the second part, we observe the effects of c on the average reward. In this example, the average reward per day is \$299.2, i.e., $g^*=299.2$, when $c=2$. The average reward increases to \$311.5 per day as a result of increasing c from 2 to 3 which means the additional average reward per day is \$12.3. Consequently, it is reasonable to increase the number of servers unless the cost of a new server is more than \$12.3.

As a further numerical study, we increase the number of servers from 1 to 10, and observe the effect of changes in c on the average reward by assuming that the cost of a new server is \$5 per day. Then, the optimal number of servers is 3 since the additional reward that the 4th server can bring is only *1.6<5* (See Table 2).

Table 2 Average reward vs. number of servers

# of servers	Average reward ($)
1	226.3
2	299.2
3	311.5
4	313.1
5	313.4
6	313.4
7	313.4
8	313.4
9	313.4
10	313.4

6. CONCLUSION

In this paper, we observe the effect of changes in the system parameters on an M/M/c queuing system and the dynamic pricing policies. We first establish the monotonicity and the concavity of the relative value function and the monotonicity of the optimal dynamic pricing policy. Then, we work on the main contribution of this study, sensitivity analysis. In this analysis, we state that the optimal prices decrease when either the service rate or the number of servers increases. On the other hand, the optimal prices decrease when the arrival rate increases. Finally, in a numerical example, we illustrate our results and the monotonicity and the concavity of the average reward that we do not discuss theoretically.

For future research perspectives, we will concentrate on further research on the sensitivity analysis in the stochastic control problems. One interesting direction is to perform sensitivity analysis on different control policies, such as admission control and controlled departure. We are currently working on generalizing our findings in this study.

In conclusion, we mainly focused on finding structural results about the effects of the parameters on the system and these results can be used in many kinds of real applications where dynamic pricing is employed. Telecommunication systems, call centers, tolls in high ways are some of the systems in which our results can be used.

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