Sensitivity Analysis on a Dynamic Pricing Problem of an M/M/c Queueing System

(Technical Appendix)

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In this technical appendix, we show the complete proofs of lemmas and theorems introduced in [1]. To simplify the notation, we define $\Delta u(x)$ such that $\Delta u(x) = u(x) - u(x+1)$. Furthermore, while considering the effects of parameters, we denote the value function of the system i with the parameter α_i, the parameter that we would like to observe, by $u^i(x)$.

1 Lemma 1 (Monotonicity)

In this proof, we only show the monotonicity of $T_P u(x)$ because monotonicity of $T_D u(x)$ has already been proven by Koole [2] and $T_F u(x)$ is the same as $u(x)$. We let p_x be the optimal price for the state x and then prove that if $u(x)$ is non-increasing in x then, the following equation will be true.

$$T_P u(x) \geq T_P u(x+1). \tag{1}$$

We can rewrite Equation 1 according to the optimality equation as:

$$\bar{F}_R(p_x)[u(x+1)+p_x] + F_R(p_x)u(x) \geq \bar{F}_R(p_{x+1})[u(x+2)+p_{x+1}] + F_R(p_{x+1})u(x+1). \tag{2}$$

Since p_x is the optimal price for the state x, we have that,

$$\bar{F}_R(p_x)[u(x+1)+p_x] + F_R(p_x)u(x) \geq \bar{F}_R(p_{x+1})[u(x+1)+p_{x+1}] + F_R(p_{x+1})u(x). \tag{3}$$

Moreover, as a result of the monotonicity of $u(x)$,

$$\bar{F}_R(p_{x+1})[u(x+1)+p_{x+1}] + F_R(p_{x+1})u(x) \geq \bar{F}_R(p_{x+1})[u(x+2)+p_{x+1}] + F_R(p_{x+1})u(x+1). \tag{4}$$

Then, it is obvious that Equation 2 is true when we combine Equations 3 and 4. Thus, the proof is completed and $T_P u(x)$ is non-increasing in x if $u(x)$ is non-increasing in x.

2 Lemma 2 (Concavity)

As in the monotonicity proof, we only focus on the concavity preserved by the pricing operator. In other words, we show that the following equation is true under the assumption that $u(x)$ is non-increasing and concave in x.

$$\Delta T_P u(x) \leq \Delta T_P u(x+1). \tag{5}$$

We let p_x be the optimal price for the state x and then rewrite Equation 5 by using the optimality equation as:

$$\begin{array}{c} \bar{F}_R(p_x)[u(x+1)+p_x] + F_R(p_x)u(x) \\ -\bar{F}_R(p_{x+1})[u(x+2)+p_{x+1}] - F_R(p_{x+1})u(x+1) \end{array} \leq \begin{array}{c} \bar{F}_R(p_{x+1})[u(x+2)+p_{x+1}] + F_R(p_{x+1})u(x+1) \\ -\bar{F}_R(p_{x+2})[u(x+3)+p_{x+2}] - F_R(p_{x+2})u(x+2). \end{array} \tag{6}$$

We first concentrate on the left hand side of Equation 6. As a result of the optimality of p_{x+1} for the state $x+1$, we have that,

$$\begin{array}{c} \bar{F}_R(p_x)[u(x+1)+p_x] + F_R(p_x)u(x) \\ -\bar{F}_R(p_{x+1})[u(x+2)+p_{x+1}] - F_R(p_{x+1})u(x+1) \end{array} \leq \begin{array}{c} \bar{F}_R(p_x)[u(x+1)+p_x] + F_R(p_x)u(x) \\ -\bar{F}_R(p_x)[u(x+2)+p_x] - F_R(p_x)u(x+1). \end{array}$$

After rearranging the right hand side of the above equation,

$$\begin{array}{c} \bar{F}_R(p_x)[u(x+1)+p_x] + F_R(p_x)u(x) \\ -\bar{F}_R(p_{x+1})[u(x+2)+p_{x+1}] - F_R(p_{x+1})u(x+1) \end{array} \leq \begin{array}{c} [u(x+1)-u(x+2)] \\ F_R(p_x)[[u(x)-u(x+1)] - [u(x+1)-u(x+2)]]. \end{array}$$

Since we assume the concavity of $u(x)$, we also have that,

$$\begin{array}{c} \bar{F}_R(p_x)[u(x+1)+p_x] + F_R(p_x)u(x) \\ -\bar{F}_R(p_{x+1})[u(x+2)+p_{x+1}] - F_R(p_{x+1})u(x+1) \end{array} \leq u(x+1) - u(x+2). \tag{7}$$

Similarly, by concentrating on the right hand side of Equation 6 and using the optimality of p_{x+1}, we obtain that,

$$\begin{array}{c} \bar{F}_R(p_{x+1})[u(x+2)+p_{x+1}] + F_R(p_{x+1})u(x+1) \\ -\bar{F}_R(p_{x+2})[u(x+3)+p_{x+2}] - F_R(p_{x+2})u(x+2) \end{array} \geq \begin{array}{c} \bar{F}_R(p_{x+2})[u(x+2)+p_{x+2}] + F_R(p_{x+2})u(x+1) \\ -\bar{F}_R(p_{x+2})[u(x+3)+p_{x+2}] - F_R(p_{x+2})u(x+2). \end{array}$$

Then, if we rearrange the right hand side of this equation due to the concavity of $u(x)$, we will have,

$$\begin{array}{c} \bar{F}_R(p_{x+1})[u(x+2)+p_{x+1}] + F_R(p_{x+1})u(x+1) \\ -\bar{F}_R(p_{x+2})[u(x+3)+p_{x+2}] - F_R(p_{x+2})u(x+2) \end{array} \geq u(x+1) - u(x+2). \tag{8}$$

Finally, when we combine Equations 7 and 8, it is obvious that Equation 6 is true. Thus, $T_P u(x)$ preserves concavity of $u(x)$ in x.

3 Lemma 3 (Submodularity)

Assume that $u(x)$ is non-increasing and concave in x and let α be the parameter whose effects we would like to observe. Under this assumption, we prove that a certain operator, T, preserves submodularity of $u(x)$ with respect to α and x. To achieve this aim, we show that Equation 9 is true for a certain operator, T. As we mentioned in [1], we do not investigate the submodularity of $Tu(x)$ with respect to the number of servers and x, i.e., the parameter α can be either the service rate, μ, or the arrival, λ. The proof for the fictitious operator is omitted because it is the same as the value function and thus, it is obvious that $T_F u(x)$ preserves all of the structural properties of $u(x)$.

$$\Delta T u^1(x) \leq \Delta T u^2(x) \tag{9}$$

3.1 The Departure Operator

Since $T_D u(x)$ is defined as a piecewise function, we have to observe the submodularity for both the cases $x < c$ and $x \geq c$. For $x < c$, we can rewrite Equation 9 for the departure operator as:

$$\begin{array}{ccc} \frac{x}{M}u^1(x-1) + \left(1 - \frac{x}{M}\right)u^1(x) & & \frac{x}{M}u^2(x-1) + \left(1 - \frac{x}{M}\right)u^2(x) \\ -\frac{x+1}{M}u^1(x) - \left(1 - \frac{x+1}{M}\right)u^1(x+1) & \leq & -\frac{x+1}{M}u^2(x) - \left(1 - \frac{x+1}{M}\right)u^2(x+1). \end{array} \tag{10}$$

When we rearrange this equation, we obtain that,

$$\begin{array}{ccc} \frac{x}{M}[u^1(x-1) - u^1(x)] & & \frac{x}{M}[u^2(x-1) - u^2(x)] \\ \left(1 - \frac{x+1}{M}\right)[u^1(x) - u^1(x+1)] & \leq & \left(1 - \frac{x+1}{M}\right)[u^2(x) - u^2(x+1)]. \end{array}$$

It is obvious that this equation is true as a result of the submodularity of $u(x)$. Therefore, Equation 10 is also true and we show that the submodularity of $T_D u(x)$ for $x < c$. For $x \geq c$, we can rewrite the submodularity equation, Equation 9, for the departure operator as:

$$\begin{array}{ccc} \frac{c}{M}u^1(x-1) + \left(1 - \frac{c}{M}\right)u^1(x) & & \frac{c}{M}u^2(x-1) + \left(1 - \frac{c}{M}\right)u^2(x) \\ -\frac{c}{M}u^1(x) - \left(1 - \frac{c}{M}\right)u^1(x+1) & \leq & -\frac{c}{M}u^2(x) - \left(1 - \frac{c}{M}\right)u^2(x+1), \end{array} \tag{11}$$

and as in the case $x < c$, when we rearrange this equation, we have that,

$$\begin{array}{ccc} \frac{c}{M}[u^1(x-1) - u^1(x)] & & \frac{c}{M}[u^2(x-1) - u^2(x)] \\ \left(1 - \frac{c}{M}\right)[u^1(x) - u^1(x+1)] & \leq & \left(1 - \frac{c}{M}\right)[u^2(x) - u^2(x+1)]. \end{array}$$

This equation is true by the submodularity of $u(x)$ and thus, Equation 11 is also true. Therefore, we show the submodularity of $T_D u(x)$ for $x \geq c$ and complete the proof.

3.2 The Pricing Operator

Let p_x^i be the optimal price for the state x in system i. Then, we can write the submodularity equation as:

$$\begin{array}{c} \bar{F}_R(p_x^1)[u^1(x+1)+p_x^1]+F_R(p_x^1)u^1(x) \\ -\bar{F}_R(p_{x+1}^1)[u^1(x+2)+p_{x+1}^1]-F_R(p_{x+1}^1)u^1(x+1) \end{array} \leq \begin{array}{c} \bar{F}_R(p_x^2)[u^2(x+1)+p_x^2]+F_R(p_x^2)u^2(x) \\ -\bar{F}_R(p_{x+1}^2)[u^2(x+2)+p_{x+1}^2]-F_R(p_{x+1}^2)u^2(x+1). \end{array} \tag{12}$$

As in the concavity proof, we first consider the left hand side of Equation 12. Then, by using the optimality of p_{x+1}^1, we have that,

$$\begin{array}{c} \bar{F}_R(p_x^1)[u^1(x+1)+p_x^1]+F_R(p_x^1)u^1(x) \\ -\bar{F}_R(p_{x+1}^1)[u^1(x+2)+p_{x+1}^1]-F_R(p_{x+1}^1)u^1(x+1) \end{array} \leq \begin{array}{c} \bar{F}_R(p_x^1)[u^1(x+1)+p_x^1]+F_R(p_x^1)u^1(x) \\ -\bar{F}_R(p_{x+1}^2)[u^1(x+2)+p_{x+1}^2]-F_R(p_{x+1}^2)u^1(x+1). \end{array}$$

If we rearrange the right hand side of the above equation, we can obtain,

$$\begin{array}{c} \bar{F}_R(p_x^1)[u^1(x+1)+p_x^1]+F_R(p_x^1)u^1(x) \\ -\bar{F}_R(p_{x+1}^1)[u^1(x+2)+p_{x+1}^1]-F_R(p_{x+1}^1)u^1(x+1) \end{array} \leq \begin{array}{c} \bar{F}_R(p_x^1)p_x^1-\bar{F}_R(p_{x+1}^2)p_{x+1}^2 \\ +F_R(p_x^1)[u^1(x)-u^1(x+1)] \\ +\bar{F}_R(p_{x+1}^2)[u^1(x+1)-u^1(x+2)]. \end{array} \tag{13}$$

Similarly, we work on the right hand side of Equation 12 and obtain the following equation by using the optimality of p_x^2:

$$\begin{array}{c} \bar{F}_R(p_x^2)[u^2(x+1)+p_x^2]+F_R(p_x^2)u^2(x) \\ -\bar{F}_R(p_{x+1}^2)[u^2(x+2)+p_{x+1}^2]-F_R(p_{x+1}^2)u^2(x+1) \end{array} \geq \begin{array}{c} \bar{F}_R(p_x^1)p_x^1-\bar{F}_R(p_{x+1}^2)p_{x+1}^2 \\ +F_R(p_x^1)[u^1(x)-u^1(x+1)] \\ +\bar{F}_R(p_{x+1}^2)[u^1(x+1)-u^1(x+2)]. \end{array} \tag{14}$$

When we combine Equations 13 and 14, it is obvious that the submodularity equation for this operator, Equation 12, is true and thus, we complete the proof of the submodularity of $T_P u(x)$ with respect to α and x.

4 Lemma 4 (Supermodularity)

In this proof, we show that the following equation is true for operators: T_D and T_P under the assumptions that $u(x)$ is non-increasing and concave in x and supermodular with respect to α and x. Unlike the submodularity proof, we also consider the number of servers as a parameter that we would like to observe. We omit the proof for the fictitious operator since it it the same as $u(x)$.

$$\Delta T u^1(x) \geq \Delta T u^2(x) \tag{15}$$

4.1 The Departure Operator

As we mentioned before, $T_D u(x)$ is defined as a piecewise function depending on the number of servers, c, so that, any changes in c also changes the definition of the operator. Therefore, we have to analyze the proof first considering the parameter α as μ or λ and then, considering α as c. While considering μ or λ, the proof of the supermodularity of the operator is similar to the proof of the submodularity: We examine the supermodularity equation both for $x < c$ and $x \geq c$. For $x < c$, we can write Equation 15 as:

$$\begin{array}{ccc} \frac{x}{M}u^1(x-1) + \left(1 - \frac{x}{M}\right)u^1(x) & & \frac{x}{M}u^2(x-1) + \left(1 - \frac{x}{M}\right)u^2(x) \\ -\frac{x+1}{M}u^1(x) - \left(1 - \frac{x+1}{M}\right)u^1(x+1) & \geq & -\frac{x+1}{M}u^2(x) - \left(1 - \frac{x+1}{M}\right)u^2(x+1), \end{array}$$

and when we rearrange this equation, we obtain that,

$$\begin{array}{ccc} \frac{x}{M}[u^1(x-1) - u^1(x)] & & \frac{x}{M}[u^2(x-1) - u^2(x)] \\ \left(1 - \frac{x+1}{M}\right)[u^1(x) - u^1(x+1)] & \geq & \left(1 - \frac{x+1}{M}\right)[u^2(x) - u^2(x+1)]. \end{array} \tag{16}$$

It is obvious that, Equation 16 is true by the supermodularity of $u(x)$. For $x \geq c$, Equation 15 can be written and rearranged as:

$$\begin{array}{ccc} \frac{c}{M}[u^1(x-1) - u^1(x)] & & \frac{c}{M}[u^2(x-1) - u^2(x)] \\ \left(1 - \frac{c}{M}\right)[u^1(x) - u^1(x+1)] & \geq & \left(1 - \frac{c}{M}\right)[u^2(x) - u^2(x+1)], \end{array}$$

and it is also true by the supermodularity of $u(x)$. Thus, we prove that the departure operator preserves the supermodularity of $u(x)$ with respect to α and x, where α is either μ or λ.

Now, we examine the submodularity of $T_D u(x)$ with respect to c and x. Let c_i be the number of servers in system i and assume that $c_2 = c_1 + \varepsilon$, such that $\varepsilon \in \mathbb{Z}^+$. Then, we show that the supermodularity equation, Equation 15, is true for $x < c_1$, $c_1 \leq x < c_2$, and $x \geq c_2$. The proof of Equation 15 for $x < c_1$, and $x \geq c_2$ can be shown by the same reasonings that we discuss while considering the service and arrival rates. However, we have to investigate the case $c_1 \leq x < c_2$ to complete the proof. For $c_1 \leq x < c_2$, Equation 15 can be written as follows by letting $x = c_1 + a$, where $0 \leq a < \varepsilon$:

$$\begin{array}{ccc} \frac{c_1}{M}u^1(x-1) + \left(1 - \frac{c_1}{M}\right)u^1(x) & & \frac{c_1+a}{M}u^2(x-1) + \left(1 - \frac{c_1+a}{M}\right)u^2(x) \\ -\frac{c_1}{M}u^1(x) - \left(1 - \frac{c_1}{M}\right)u^1(x+1) & \geq & -\frac{c_1+a+1}{M}u^2(x) - \left(1 - \frac{c_1+a+1}{M}\right)u^2(x+1). \end{array} \tag{17}$$

When we rearrange Equation 17, we have that,

$$\begin{array}{c}\frac{c_1}{M}[u^1(x-1)-u^1(x)]\\ +\left(1-\frac{c_1}{M}\right)[u^1(x)-u^1(x+1)]\end{array} \geq \begin{array}{c}\frac{c_1}{M}[u^2(x-1)-u^2(x)]\\ +\left(1-\frac{c_1}{M}\right)[u^2(x)-u^2(x+1)]\\ \frac{a}{M}[[u^2(x-1)-u^2(x)]-[u^2(x)-u^2(x+1)]]\\ +u^2(x+1)-u^2(x)\end{array}.$$

In this equation, first two lines are satisfied by the supermodularity of $u(x)$ and last two lines are true as a result of the concavity and the monotonicity of $u(x)$, respectively. Hence, we show that $T_D u(x)$ is also submodular with respect to c and x and complete the proof.

4.2 The Pricing Operator

This proof is similar to the proof of the submodularity of $u(x)$. We let p_x^i be the optimal price for the state x in system i and then, show that the following equation is true under given assumptions.

$$\begin{array}{c}\bar{F}_R(p_x^1)[u^1(x+1)+p_x^1]+F_R(p_x^1)u^1(x)\\ -\bar{F}_R(p_{x+1}^1)[u^1(x+2)+p_{x+1}^1]-F_R(p_{x+1}^1)u^1(x+1)\end{array} \geq \begin{array}{c}\bar{F}_R(p_x^2)[u^2(x+1)+p_x^2]+F_R(p_x^2)u^2(x)\\ -\bar{F}_R(p_{x+1}^2)[u^2(x+2)+p_{x+1}^2]-F_R(p_{x+1}^2)u^2(x+1).\end{array} \tag{18}$$

When we concentrate on the left hand side of Equation 18, we have the following equation by using the optimality of p_x^1:

$$\begin{array}{c}\bar{F}_R(p_x^1)[u^1(x+1)+p_x^1]+F_R(p_x^1)u^1(x)\\ -\bar{F}_R(p_{x+1}^1)[u^1(x+2)+p_{x+1}^1]-F_R(p_{x+1}^1)u^1(x+1)\end{array} \geq \begin{array}{c}\bar{F}_R(p_x^2)p_x^2-\bar{F}_R(p_{x+1}^1)p_{x+1}^1\\ +F_R(p_x^2)[u^1(x)-u^1(x+1)]\\ +\bar{F}_R(p_{x+1}^1)[u^1(x+1)-u^1(x+2)].\end{array} \tag{19}$$

Similarly, by concentrating on the right hand side of Equation 18 and using the optimality of p_{x+1}^2, we have that,

$$\begin{array}{c}\bar{F}_R(p_x^2)[u^2(x+1)+p_x^2]+F_R(p_x^2)u^2(x)\\ -\bar{F}_R(p_{x+1}^2)[u^2(x+2)+p_{x+1}^2]-F_R(p_{x+1}^2)u^2(x+1)\end{array} \leq \begin{array}{c}\bar{F}_R(p_x^2)p_x^2-\bar{F}_R(p_{x+1}^1)p_{x+1}^1\\ +F_R(p_x^2)[u^1(x)-u^1(x+1)]\\ +\bar{F}_R(p_{x+1}^1)[u^1(x+1)-u^1(x+2)].\end{array} \tag{20}$$

When we combine Equations 19 and 20, it is obvious that Equation 18 is true and thus, we complete the proof.

5 Lemma 5 (Monotonicity of $Tu(x) - u(x)$)

In this section, we show the monotonicity of the extra gain obtained for a certain operator, $Tu(x) - u(x)$, by an increase in the service and arrival rates under the assumption that $u(x)$ is non-increasing and concave in x.

5.1 Monotonicity of $T_D u(x) - u(x)$

As we mentioned in [1], the extra gain obtained for the departure operator is non-decreasing in x. In order to prove this statement, we show that the following equation is true if $u(x)$ is non-increasing and concave in x.

$$T_D u(x) - u(x) \leq T_D u(x+1) - u(x+1) \tag{21}$$

For $x < c$, we can rewrite Equation 21 as:

$$\frac{x}{M} u(x-1) + \left(1 - \frac{x}{M}\right) u(x) - u(x) \leq \frac{x+1}{M} u(x) + \left(1 - \frac{x+1}{M}\right) u(x+1) - u(x+1).$$

After rearranging this equation, we obtain that,

$$\frac{x}{M}[u(x-1) - u(x)] \leq \frac{x+1}{M}[u(x) - u(x+1)],$$

and this equation holds by the monotonicity and concavity of $u(x)$. Moreover, while considering $x \geq c$, Equation 21 can be written and rearranged as:

$$\frac{c}{M}[u(x-1) - u(x)] \leq \frac{c}{M}[u(x) - u(x+1)],$$

which is also true by the concavity of $u(x)$. Hence, we prove that Equation 21 is true for all states x and $T_D u(x) - u(x)$ is non-decreasing in x if $u(x)$ is non-increasing and concave in x.

5.2 Monotonicity of $T_P u(x) - u(x)$

Since the extra gain obtained for the pricing operator is non-increasing in the number of customers in the system, we show that Equation 22 is true under the given assumptions.

$$T_P u(x) - u(x) \geq T_P u(x+1) - u(x+1) \tag{22}$$

We let p_x is the optimal price for the state x and then, rewrite Equation 22 as:

$$\bar{F}_R(p_x)[u(x+1) + p_x] + F_R(p_x) u(x) - u(x) \geq \bar{F}_R(p_{x+1})[u(x+2) + p_{x+1}] + F_R(p_{x+1}) u(x+1) - u(x+1). \tag{23}$$

By using the optimality of p_x, we have,

$$\bar{F}_R(p_x)[u(x+1)+p_x]+F_R(p_x)u(x)-u(x) \geq \bar{F}_R(p_{x+1})[u(x+1)+p_{x+1}]+F_R(p_{x+1})u(x)-u(x).$$

When we rearrange the right hand side,

$$\bar{F}_R(p_x)[u(x+1)+p_x]+F_R(p_x)u(x)-u(x) \geq \bar{F}_R(p_{x+1})[u(x+1)-u(x)]+\bar{F}_R(p_{x+1})p_{x+1}. \quad (24)$$

Since we assume that $u(x)$ is concave in x, we have that,

$$\bar{F}_R(p_{x+1})[u(x+1)-u(x)]+\bar{F}_R(p_{x+1})p_{x+1} \geq \bar{F}_R(p_{x+1})[u(x+2)-u(x+1)]+\bar{F}_R(p_{x+1})p_{x+1}, \quad (25)$$

and

$$\bar{F}_R(p_{x+1})[u(x+2)-u(x+1)]+\bar{F}_R(p_{x+1})p_{x+1} = \bar{F}_R(p_{x+1})[u(x+2)+p_{x+1}]+F_R(p_{x+1})u(x+1)-u(x+1). \quad (26)$$

If we combine Equations 24, 25, and 26, Equation 23 will be true and thus we complete the proof.

6 Theorem 1

6.1 Monotonicity and Concavity of $u(x)$

As we mentioned in the paper [1], we use the finite horizon problem to prove the structural properties of $u(x)$, the value function of the infinite horizon problem. Therefore, we first prove that $u_n(x)$, the expected total discounted profit of a system starting in state x with n transitions remaining in the future, is non-increasing and concave in x for all finite n by induction in order to prove the monotonicity and concavity of $u(x)$. The initial condition of the induction holds by the specification that $u_0(x)=0$ for all states x. Then, we assumed that $u_n(x) \geq u_n(x+1)$ and $\Delta u_n(x) \leq \Delta u_n(x+1)$ and show:

$$\begin{aligned} u_{n+1}(x) &\geq u_{n+1}(x+1) && (27)\\ \Delta u_{n+1}(x) &\leq \Delta u_{n+1}(x+1) && (28) \end{aligned}$$

We can rewrite Equation 27 as follows by using the optimality equation:

$$\begin{matrix} \mu M T_D u_n(x) \\ +\lambda T_P u_n(x) \\ +\theta T_F u_n(x) \\ -hx \end{matrix} \geq \begin{matrix} \mu M T_D u_n(x+1) \\ +\lambda T_P u_n(x+1) \\ +\theta T_F u_n(x+1) \\ -h(x+1) \end{matrix}.$$

The last line is obviously true and the first three lines are also true by Lemma 1. Therefore $u_n(x)$ is non-increasing in x for all finite n.

Similarly, we can rewrite Equation 28 as:

$$\begin{array}{rcl} \mu M \Delta T_D u_n(x) & & \mu M \Delta T_D u_n(x+1) \\ +\lambda \Delta T_P u_n(x) & \leq & +\lambda \Delta T_P u_n(x+1) \\ +\theta \Delta T_F u_n(x) & & +\theta \Delta T_F u_n(x+1). \end{array}$$

This equation is true by Lemma 2 and then, $u_n(x)$ is concave in x for all finite n. As we mentioned in the paper [1], structural results obtained for $u_n(x)$ are also valid for $u(x)$ and $u'(x)$, the relative value function for the average reward criterion so that, $u(x)$ and $u'(x)$ are non-decreasing and concave in x.

6.2 Monotonicity of the Optimal Price, $p^*(x)$

The concavity of the value function, $u(x)$, implies that the opportunity cost of a new customer is non-decreasing in x and thus, optimal prices are also non-decreasing in x. We prove the monotonicity of the optimal price, i.e., $p^*(x) \leq p^*(x+1)$, by contradiction. We let p_x be the optimal price for the state x and assume that $p_x > p_{x+1}$, which implies that $F_R(p_{x+1}) < F_R(p_x)$. Then, we have the following equations as result of the optimality of p_x and p_{x+1}.

$$\bar{F}_R(p_x)[u(x+1)+p_x] + F_R(p_x)u(x) \geq \bar{F}_R(p_{x+1})[u(x+1)+p_{x+1}] + F_R(p_{x+1})u(x),$$
$$\bar{F}_R(p_{x+1})[u(x+2)+p_{x+1}] + F_R(p_{x+1})u(x+1) \geq \bar{F}_R(p_x)[u(x+2)+p_x] + F_R(p_x)u(x+1).$$

When we sum up all the terms in these equations:

$$[F_R(p_x) - F_R(p_{x+1})][[u(x) - u(x+1)] - [u(x+1) - u(x+2)]] \geq 0. \tag{29}$$

Since $u(x)$ is concave in x and $F_R(p_{x+1}) < F_R(p_x)$ as a result of our assumption on the optimal prices, $p_x > p_{x+1}$, Equation 29 can not be true and there is a contradiction. Thus, the optimal price for the state x, $p^*(x)$, should be non-decreasing in x, i.e., $p^*(x) \leq p^*(x+1)$.

7 Theorem 2

7.1 Supermodularity of $u(x)$

As we did in Theorem 1, we first prove the supermodularity of $u_n(x)$ with respect to for all finite n by induction and the initial condition of the induction is true by the specification on $u_0(x)$. Then, we assume that $u_n(x)$ is supermodular with respect to μ and x, and show that $u_{n+1}(x)$ is also supermodular with respect to μ and x. We write the supermodularity of $u_{n+1}(x)$ as:

$$\begin{array}{ccc} \mu M \Delta T_D u_n^1(x) & & \mu M \Delta T_D u_n^2(x) \\ +\lambda \Delta T_P u_n^1(x) & \geq & +\lambda \Delta T_P u_n^2(x) \\ +\theta \Delta T_F u_n^1(x) & & +\theta \Delta T_F u_n^2(x) \\ & & +\varepsilon M[\Delta T_D u_n^2(x) - \Delta T_F u_n^2(x)]. \end{array}$$

The first three lines of this equation are true by Lemma 4 and the last line is true by Lemma 5. Therefore, $u_n(x)$ is supermodular with respect to μ and x for all finite x. As in Theorem 1, $u(x)$ and $u'(x)$ is also supermodular with respect to μ and x.

7.2 Effects of μ on the Optimal Price

As in the proof of the monotonicity of the optimal prices, we use the proof by contradiction technique to show that $p^*(x)$ is non-increasing in μ. We let p_x^i be the optimal price for the state x in system i and assume that $p_x^1 < p_x^2$. Then, we have that,

$$\bar{F}_R(p_x^1)[u^1(x+1) + p_x^1] + F_R(p_x^1)u^1(x) \geq \bar{F}_R(p_x^2)[u^1(x+1) + p_x^2] + F_R(p_x^2)u^1(x),$$
$$\bar{F}_R(p_x^2)[u^2(x+1) + p_x^2] + F_R(p_x^2)u^2(x) \geq \bar{F}_R(p_x^1)[u^2(x+1) + p_x^1] + F_R(p_x^1)u^2(x).$$

When we sum up all the terms in these equations, we obtain that,

$$[F_R(p_x^1) - F_R(p_x^2)][[u^1(x) - u^1(x+1)] - [u^2(x) - u^2(x+1)]] \geq 0. \quad (30)$$

However, Equation 30 can not be true because of the supermodularity of $u(x)$ and the assumption that $p_x^1 < p_x^2$. Therefore, there is a contradiction and the optimal price should be non-increasing in μ.

The proofs for Theorems 3 and 4 are similar to this proof. For the effects of λ and c on the value function, we first prove the structural properties for the finite horizon problem and then extend the results for the infinite horizon problem. On the other hand, we use contradiction to prove the effects of λ and c on the optimal prices.

References

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