

# Calculus: a complete course / Robert A. Adams

## 42 PRELIMINARIES

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In Exercises 19–24,  $f$  refers to the function with domain  $[0, 2]$  and range  $[0, 1]$ , whose graph is shown in Figure P.62. Sketch the graphs of the indicated functions and specify their domains and ranges.

19.  $2f(x)$                       20.  $-(1/2)f(x)$   
 21.  $f(2x)$                       22.  $f(x/3)$   
 23.  $1 + f(-x/2)$               24.  $2f((x-1)/2)$

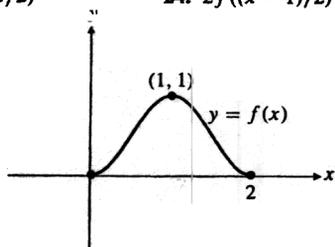


Figure P.62

In Exercises 25–26, sketch the graphs of the given functions.

25.  $f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 2-x & \text{if } 1 < x \leq 2 \end{cases}$   
 26.  $g(x) = \begin{cases} \sqrt{x} & \text{if } 0 \leq x \leq 1 \\ 2-x & \text{if } 1 < x \leq 2 \end{cases}$   
 27. Find all real values of the constants  $A$  and  $B$  for which the function  $F(x) = Ax + B$  satisfies:  
 (a)  $F \circ F(x) = F(x)$  for all  $x$ .  
 (b)  $F \circ F(x) = x$  for all  $x$ .

### Greatest and least integer functions

28. For what values of  $x$  is (a)  $\lfloor x \rfloor = 0$ ? (b)  $\lceil x \rceil = 0$ ?  
 29. What real numbers  $x$  satisfy the equation  $\lfloor x \rfloor = \lceil x \rceil$ ?  
 30. True or false:  $\lceil -x \rceil = -\lfloor x \rfloor$  for all real  $x$ ?  
 31. Sketch the graph of  $y = x - \lfloor x \rfloor$ .

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32. Sketch the graph of the function

$$f(x) = \begin{cases} \lfloor x \rfloor & \text{if } x \geq 0 \\ \lceil x \rceil & \text{if } x < 0. \end{cases}$$

(a) (b)

Why is  $f(x)$  called the integer part of  $x$ ?

### Even and odd functions

33. Assume that  $f$  is an even function,  $g$  is an odd function, and both  $f$  and  $g$  are defined on the whole real line  $\mathbb{R}$ . Is each of the following functions even, odd, or neither?

$$f + g, \quad fg, \quad f/g, \quad g/f, \quad f^2 = ff, \quad g^2 = gg$$

$$f \circ g, \quad g \circ f, \quad f \circ f, \quad g \circ g$$

34. If  $f$  is both an even and an odd function, show that  $f(x) = 0$  at every point of its domain.  
 35. Let  $f$  be a function whose domain is symmetric about the origin, that is,  $-x$  belongs to the domain whenever  $x$  does.  
 (a) Show that  $f$  is the sum of an even function and an odd function:

$$f(x) = E(x) + O(x),$$

where  $E$  is an even function and  $O$  is an odd function.  
 Hint: let  $E(x) = (f(x) + f(-x))/2$ . Show that  $E(-x) = E(x)$ , so that  $E$  is even. Then show that  $O(x) = f(x) - E(x)$  is odd.

- (b) Show that there is only one way to write  $f$  as the sum of an even and an odd function. Hint: one way is given in part (a). If also  $f(x) = E_1(x) + O_1(x)$ , where  $E_1$  is even and  $O_1$  is odd, show that  $E - E_1 = O_1 - O$  and then use Exercise 34 to show that  $E = E_1$  and  $O = O_1$ .

## P.6 The Trigonometric Functions

Most people first encounter the quantities  $\cos t$  and  $\sin t$  as ratios of sides in a right-angled triangle having  $t$  as one of the acute angles. If the sides of the triangle are labelled "hyp" for hypotenuse, "adj" for the side adjacent to angle  $t$ , and "opp" for the side opposite angle  $t$  (see Figure P.63), then

$$\cos t = \frac{\text{adj}}{\text{hyp}} \quad \text{and} \quad \sin t = \frac{\text{opp}}{\text{hyp}}$$

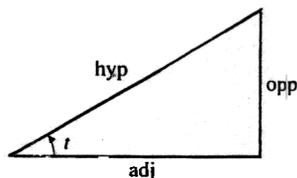


Figure P.63  $\cos t = \text{adj}/\text{hyp}$   
 $\sin t = \text{opp}/\text{hyp}$

These ratios depend only on the angle  $t$ , not on the particular triangle, since all right-angled triangles having an acute angle  $t$  are similar.

In calculus we need more general definitions of  $\cos t$  and  $\sin t$  as functions defined for all real numbers  $t$ , not just acute angles. Such definitions are phrased in terms of a circle rather than a triangle.

Let  $C$  be the circle with centre at the origin  $O$  and radius 1; its equation  $x^2 + y^2 = 1$ . Let  $A$  be the point  $(1, 0)$  on  $C$ . For any real number  $t$ , let  $P_t$  be the point on  $C$  at distance  $|t|$  from  $A$ , measured along  $C$  in the counterclockwise direction if  $t > 0$ , and the clockwise direction if  $t < 0$ . For example, since  $C$  has circumference  $2\pi$ , the point  $P_{\pi/2}$  is one-quarter of the way counterclockwise around  $C$  from  $A$ ; it is the point  $(0, 1)$ .

We will use the arc length  $t$  as a measure of the size of the angle  $AOP_t$ . See Figure P.64.

**DEFINITION 5**

The radian measure of angle  $AOP_t$  is  $t$  radians:

$$\angle AOP_t = t \text{ radians.}$$

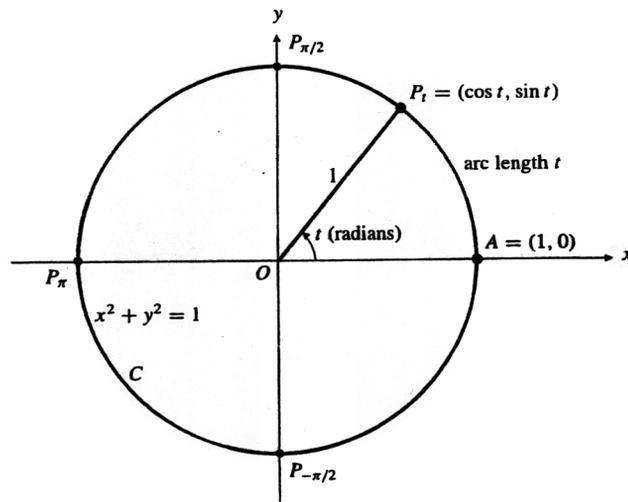


Figure P.64 If the length of arc  $AP_t$  is  $t$  units, then angle  $AOP_t = t$  radians

We are more used to measuring angles in **degrees**. Since  $P_{\pi}$  is the point  $(-1, 0)$ , halfway ( $\pi$  units of distance) around  $C$  from  $A$ , we have

$$\pi \text{ radians} = 180^\circ$$

To convert degrees to radians, multiply by  $\pi/180$ ; to convert radians to degrees, multiply by  $180/\pi$ .

**Angle convention**  
 In calculus it is assumed that all angles are measured in radians unless degrees or other units are stated explicitly. When we talk about the angle  $\pi/3$ , we mean  $\pi/3$  radians (which is  $60^\circ$ ), not  $\pi/3$  degrees.

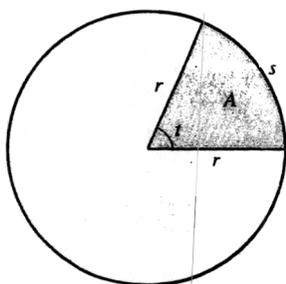


Figure P.65  
Arc length  $s = rt$   
Sector area  $A = r^2 t / 2$

**Example 1** Arc length and sector area. An arc of a circle of radius  $r$  subtends an angle  $t$  at the centre of the circle. Find the length  $s$  of the arc and the area  $A$  of the sector lying between the arc and the centre of the circle.

**Solution** The length  $s$  of the arc is the same fraction of the circumference  $2\pi r$  of the circle that the angle  $t$  is of a complete revolution  $2\pi$  radians (or  $360^\circ$ ). Thus

$$s = \frac{t}{2\pi} (2\pi r) = rt \text{ units.}$$

Similarly, the area  $A$  of the circular sector (Figure P.65) is the same fraction of the area  $\pi r^2$  of the whole circle:

$$A = \frac{t}{2\pi} (\pi r^2) = \frac{r^2 t}{2} \text{ units}^2.$$

(We will show that the area of a circle of radius  $r$  is  $\pi r^2$  in Section 1.1.)

Using the procedure described above we can find the point  $P_t$  corresponding to any real number  $t$ , positive or negative. We define  $\cos t$  and  $\sin t$  to be the coordinates of  $P_t$ . (See Figure P.66.)

### DEFINITION 6

#### Cosine and sine

For any real  $t$ , the **cosine** of  $t$  (abbreviated  $\cos t$ ) and the **sine** of  $t$  (abbreviated  $\sin t$ ) are the  $x$ - and  $y$ -coordinates of the point  $P_t$ .

$$\cos t = \text{the } x\text{-coordinate of } P_t$$

$$\sin t = \text{the } y\text{-coordinate of } P_t$$

Because they are defined this way, cosine and sine are often called the **circular functions**.

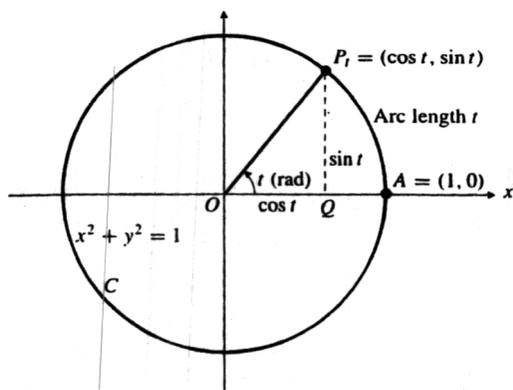


Figure P.66 The coordinates of  $P_t$  are  $(\cos t, \sin t)$

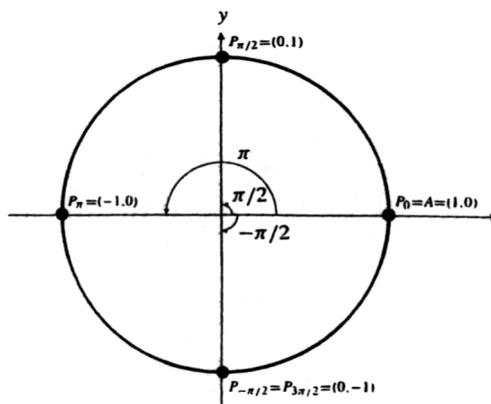


Figure P.67 Some special angles

**Example 2** Examining the coordinates of  $P_0 = A$ ,  $P_{\pi/2}$ ,  $P_\pi$ , and  $P_{-\pi/2} = P_{3\pi/2}$  in Figure P.67, we obtain the following values:

$$\begin{aligned} \cos 0 &= 1 & \cos \frac{\pi}{2} &= 0 & \cos \pi &= -1 & \cos \left(-\frac{\pi}{2}\right) &= \cos \frac{3\pi}{2} = 0 \\ \sin 0 &= 0 & \sin \frac{\pi}{2} &= 1 & \sin \pi &= 0 & \sin \left(-\frac{\pi}{2}\right) &= \sin \frac{3\pi}{2} = -1 \end{aligned}$$

### Some Useful Identities

Many important properties of  $\cos t$  and  $\sin t$  follow from the fact that they are the  $x$  and  $y$  coordinates of the point  $P_t$  on the circle  $C$  with equation  $x^2 + y^2 = 1$ .

**The range of cosine and sine.** For every real number  $t$ ,

$$-1 \leq \cos t \leq 1 \quad \text{and} \quad -1 \leq \sin t \leq 1$$

**The Pythagorean identity.** The coordinates  $x = \cos t$  and  $y = \sin t$  of the point  $P_t$  must satisfy the equation of the circle. Therefore, for every real number  $t$ ,

$$\cos^2 t + \sin^2 t = 1$$

(Note that  $\cos^2 t$  means  $(\cos t)^2$ , not  $\cos(\cos t)$ . This is an unfortunate notation, but it is used everywhere in technical literature, so you have to get used to it!)

**Periodicity.** Since  $C$  has circumference  $2\pi$ , adding  $2\pi$  to  $t$  causes the point  $P_t$  to go one extra complete revolution around  $C$  and end up in the same place  $P_{t+2\pi} = P_t$ . Thus, for every  $t$ ,

$$\cos(t + 2\pi) = \cos t \quad \text{and} \quad \sin(t + 2\pi) = \sin t$$

This says that cosine and sine are **periodic** with period  $2\pi$ .

**Cosine is an even function. Sine is an odd function.** Since the circle  $x^2 + y^2 = 1$  is symmetric about the  $x$ -axis, the points  $P_{-t}$  and  $P_t$  have the same  $x$ -coordinates and opposite  $y$ -coordinates (Figure P.68).

$$\cos(-t) = \cos t \quad \text{and} \quad \sin(-t) = -\sin t$$

**Complementary angle identities.** Two angles are complementary if their sum is  $\pi/2$  (or  $90^\circ$ ). The points  $P_{(\pi/2)-t}$  and  $P_t$  are reflections of each other in the line  $y = x$  (Figure P.69), so the  $x$ -coordinate of one is the  $y$ -coordinate of the other and vice versa. Thus,

$$\cos\left(\frac{\pi}{2} - t\right) = \sin t \quad \text{and} \quad \sin\left(\frac{\pi}{2} - t\right) = \cos t$$

**Supplementary angle identities.** Two angles are supplementary if their sum is  $\pi$  (or  $180^\circ$ ). Since the circle is symmetric about the  $y$ -axis,  $P_{\pi-t}$  and  $P_t$  have the same  $y$ -coordinates and opposite  $x$ -coordinates. (See Figure P.70.) Thus,

$\cos(\pi - t) = -\cos t$  and  $\sin(\pi - t) = \sin t$ .

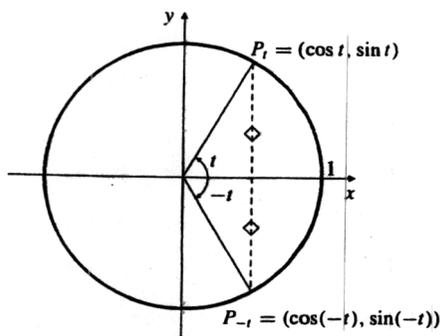


Figure P.68  $\cos(-t) = \cos t$   
 $\sin(-t) = -\sin t$

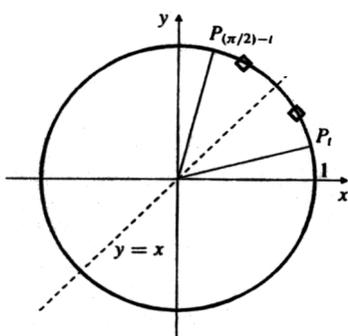


Figure P.69  $\cos((\pi/2) - t) = \sin t$   
 $\sin((\pi/2) - t) = \cos t$

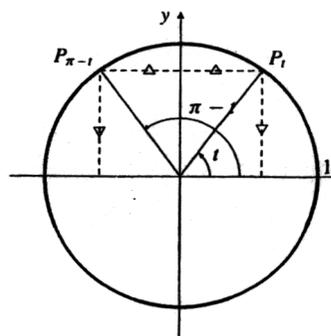


Figure P.70  $\cos(\pi - t) = -\cos t$   
 $\sin(\pi - t) = \sin t$

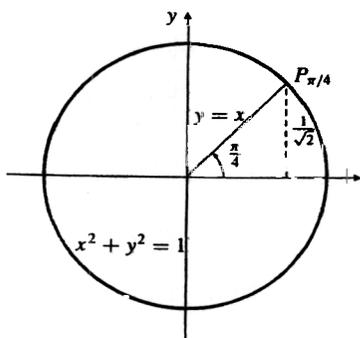


Figure P.71  $\sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

### Some Special Angles

**Example 3** Find the sine and cosine of  $\pi/4$  (that is  $45^\circ$ ).

**Solution** The point  $P_{\pi/4}$  lies in the first quadrant on the line  $x = y$ . To find its coordinates, substitute  $y = x$  into the equation  $x^2 + y^2 = 1$  of the circle, obtaining  $2x^2 = 1$ . Thus  $x = y = 1/\sqrt{2}$ , (see Figure P.71) and

$$\cos(45^\circ) = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad \sin(45^\circ) = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

**Example 4** Find the values of sine and cosine of the angles  $\pi/3$  (or  $60^\circ$ ) and  $\pi/6$  (or  $30^\circ$ ).

**Solution** The point  $P_{\pi/3}$  and the points  $O(0, 0)$  and  $A(1, 0)$  are the vertices of an equilateral triangle with edge length 1 (see Figure P.72). Thus  $P_{\pi/3}$  has  $x$ -coordinate  $1/2$  and  $y$ -coordinate  $\sqrt{1 - (1/2)^2} = \sqrt{3}/2$ , and

$$\cos(60^\circ) = \cos \frac{\pi}{3} = \frac{1}{2}, \quad \sin(60^\circ) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

Since  $\frac{\pi}{6} = \frac{\pi}{2} - \frac{\pi}{3}$ , the complementary angle identities now tell us that

$$\cos(30^\circ) = \cos \frac{\pi}{6} = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}, \quad \sin(30^\circ) = \sin \frac{\pi}{6} = \cos \frac{\pi}{3} = \frac{1}{2}$$

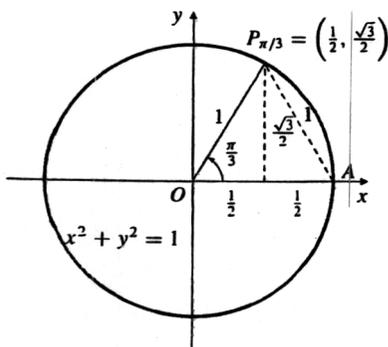


Figure P.72  $\cos \pi/3 = 1/2$   
 $\sin \pi/3 = \sqrt{3}/2$

Table 5 summarizes the values of cosine and sine at multiples of  $30^\circ$  and  $45^\circ$

between  $0^\circ$  and  $180^\circ$ . The values for  $120^\circ$ ,  $135^\circ$ , and  $150^\circ$  were determined by using the supplementary angle identities; for example,

$$\cos(120^\circ) = \cos\left(\frac{2\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\cos(60^\circ) = -\frac{1}{2}$$

Table 5. Cosines and sines of special angles

Degrees	$0^\circ$	$30^\circ$	$45^\circ$	$60^\circ$	$90^\circ$	$120^\circ$	$135^\circ$	$150^\circ$	$180^\circ$
Radians	0			$\frac{\pi}{3}$	$\frac{\pi}{2}$				$\pi$
Cosine	1		$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$		-1
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$				$\frac{\sqrt{2}}{2}$		0

**Example 5** Find: (a)  $\sin(3\pi/4)$  and (b)  $\cos(4\pi/3)$ .

**Solution** We can draw appropriate triangles in the quadrants where the angles lie to determine the required values. See Figure P.73.

$$(a) \sin(3\pi/4) = \sin(\pi - (\pi/4)) = 1/\sqrt{2}.$$

$$(b) \cos(4\pi/3) = \cos(\pi + (\pi/3)) = -\frac{1}{2}.$$

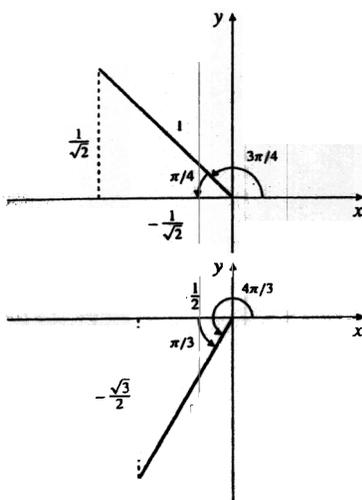


Figure P.73 Using suitably placed triangles to find trigonometric functions of special angles

While decimal approximations to the values of sine and cosine can be found using a scientific calculator or mathematical tables, it is useful to remember the exact values in the table for angles  $0$ ,  $\pi/6$ ,  $\pi/4$ ,  $\pi/3$ , and  $\pi/2$ . They occur frequently in applications.

When we treat sine and cosine as functions we can call the variable they depend on  $x$  (as we do with other functions), rather than  $t$ . The graphs of  $\cos x$  and  $\sin x$  are shown in Figures P.74 and P.75. In both graphs the pattern between  $x = 0$  and  $x = 2\pi$  repeats over and over to the left and right. Observe that the graph of  $\sin x$  is the graph of  $\cos x$  shifted to the right a distance  $\pi/2$ .

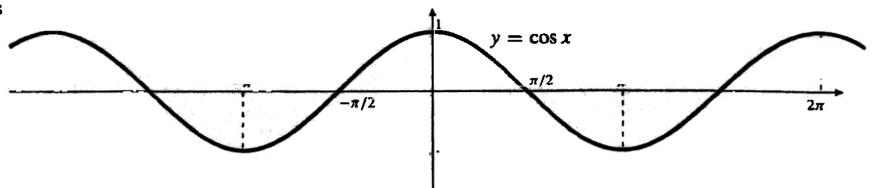


Figure P.74 The graph of  $\cos x$

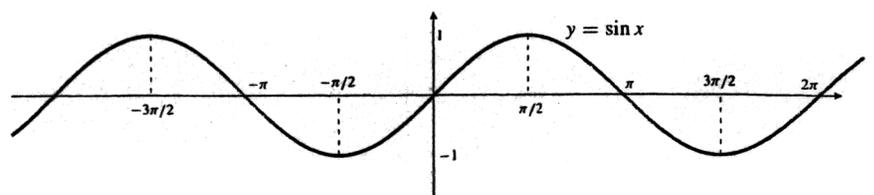


Figure P.75 The graph of  $\sin x$

**Remember this!**

When using a scientific calculator to calculate any trigonometric functions, be sure you have selected the proper angular mode: degrees or radians.

**The Addition Formulas**

The following formulas enable us to determine the cosine and sine of a sum or difference of two angles in terms of the cosines and sines of those angles.

**THEOREM****1****Addition Formulas for Cosine and Sine**

$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$

$$\sin(s + t) = \sin s \cos t + \cos s \sin t$$

$$\cos(s - t) = \cos s \cos t + \sin s \sin t$$

$$\sin(s - t) = \sin s \cos t - \cos s \sin t$$

**PROOF** We prove the third of these formulas as follows: Let  $s$  and  $t$  be real numbers and consider the points

$$P_t = (\cos t, \sin t)$$

$$P_{s-t} = (\cos(s-t), \sin(s-t))$$

$$P_s = (\cos s, \sin s)$$

$$A = (1, 0),$$

as shown in Figure P.76.

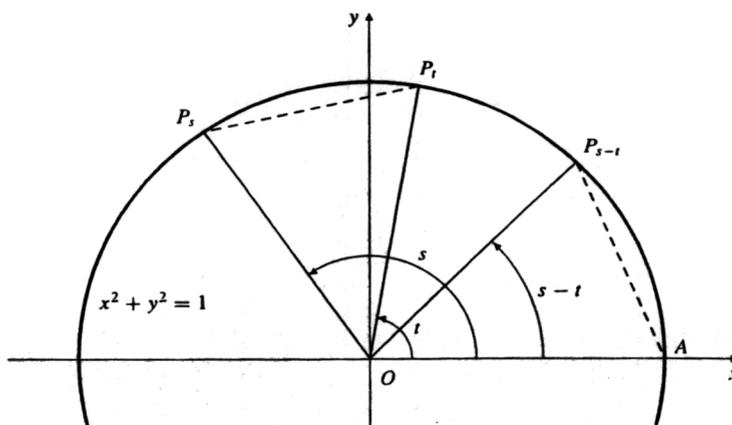


Figure P.76  $P_s P_t = P_{s-t} A$

The angle  $P_t O P_s = s - t$  radians = angle  $A O P_{s-t}$ , so the distance  $P_s P_t$  is equal to the distance  $P_{s-t} A$ . Therefore,  $(P_s P_t)^2 = (P_{s-t} A)^2$ . We express these squared distances in terms of coordinates and expand the resulting squares of binomials:

$$(\cos s - \cos t)^2 + (\sin s - \sin t)^2 = (\cos(s-t) - 1)^2 + \sin^2(s-t),$$

$$\begin{aligned} \cos^2 s - 2 \cos s \cos t + \cos^2 t + \sin^2 s - 2 \sin s \sin t + \sin^2 t \\ = \cos^2(s-t) - 2 \cos(s-t) + 1 + \sin^2(s-t). \end{aligned}$$

Since  $\cos^2 x + \sin^2 x = 1$  for every  $x$ , this reduces to

$$\cos(s - t) = \cos s \cos t + \sin s \sin t.$$

Replacing  $t$  with  $-t$  in the formula above, and recalling that  $\cos(-t) = \cos t$  and  $\sin(-t) = -\sin t$ , we have

$$\cos(s + t) = \cos s \cos t - \sin s \sin t.$$

The complementary angle formulas can be used to obtain either of the addition formulas for sine:

$$\begin{aligned} \sin(s + t) &= \cos\left(\frac{\pi}{2} - (s + t)\right) \\ &= \cos\left(\left(\frac{\pi}{2} - s\right) - t\right) \\ &= \cos\left(\frac{\pi}{2} - s\right) \cos t + \sin\left(\frac{\pi}{2} - s\right) \sin t \\ &= \sin s \cos t + \cos s \sin t, \end{aligned}$$

and the other formula again follows if we replace  $t$  with  $-t$ .

**Example 6** Find the value of  $\cos(\pi/12) = \cos 15^\circ$ .

*Solution*

$$\begin{aligned} \cos \frac{\pi}{12} &= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \sin \frac{\pi}{4} \\ &= \left(\frac{1}{2}\right) \left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2}\right) \left(\frac{1}{\sqrt{2}}\right) = \frac{1 + \sqrt{3}}{2\sqrt{2}} \end{aligned}$$

From the addition formulas, we obtain as special cases certain useful formulas called **double-angle formulas**. Put  $s = t$  in the addition formulas for  $\sin(s + t)$  and  $\cos(s + t)$  to get

$$\begin{aligned} \sin 2t &= 2 \sin t \cos t \quad \text{and} \\ \cos 2t &= \cos^2 t - \sin^2 t \\ &= 2 \cos^2 t - 1 \quad \text{(using } \sin^2 t + \cos^2 t = 1) \\ &= 1 - 2 \sin^2 t \end{aligned}$$

Solving the last two formulas for  $\cos^2 t$  and  $\sin^2 t$ , we obtain

$$\cos^2 t = \frac{1 + \cos 2t}{2} \quad \text{and} \quad \sin^2 t = \frac{1 - \cos 2t}{2},$$

which are sometimes called **half-angle formulas** because they are used to express trigonometric functions of half of the angle  $2t$ . Later we will find these formulas useful when we have to integrate powers of  $\cos x$  and  $\sin x$ .

### Other Trigonometric Functions

There are four other trigonometric functions—tangent ( $\tan$ ), cotangent ( $\cot$ ), secant ( $\sec$ ), and cosecant ( $\csc$ )—each defined in terms of cosine and sine. Their graphs are shown in Figures P.77–P.80.

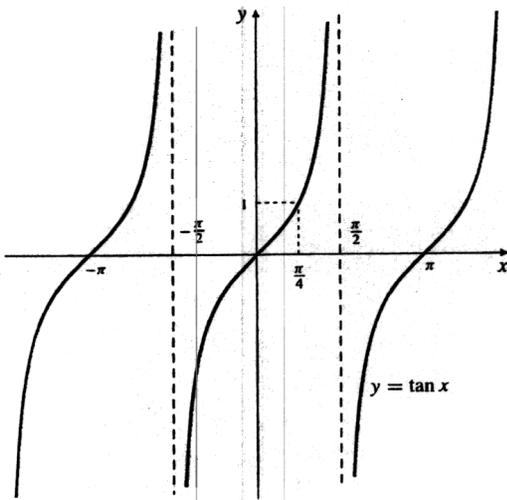


Figure P.77 The graph of  $\tan x$

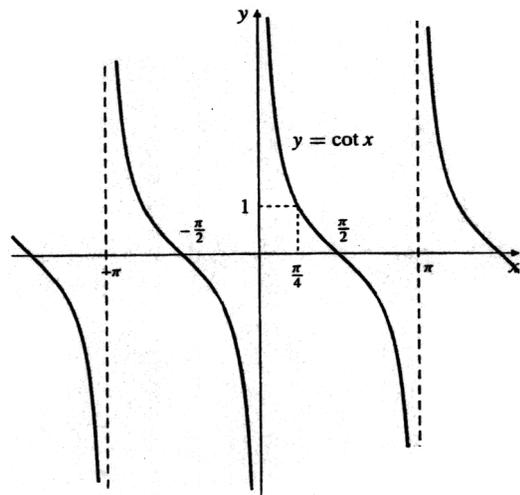


Figure P.78 The graph of  $\cot x$

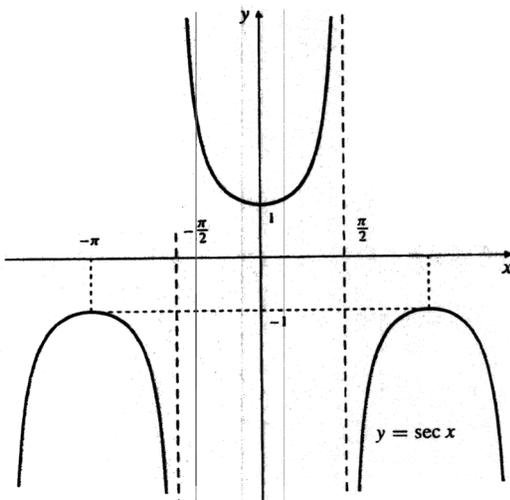


Figure P.79 The graph of  $\sec x$

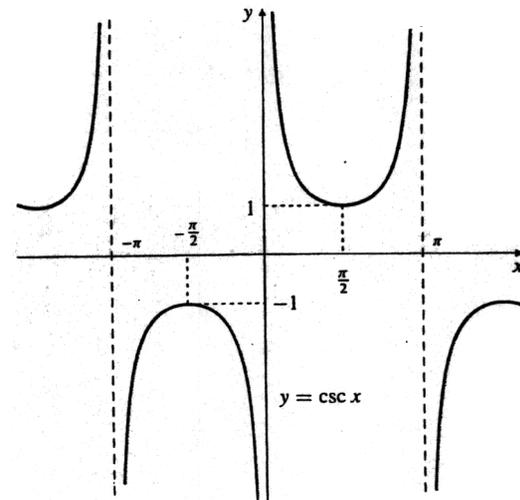


Figure P.80 The graph of  $\csc x$

**DEFINITION 7**

**Tangent, cotangent, secant, and cosecant**

$$\begin{aligned} \tan t &= \frac{\sin t}{\cos t} & \sec t &= \frac{1}{\cos t} \\ \cot t &= \frac{\cos t}{\sin t} = \frac{1}{\tan t} & \csc t &= \frac{1}{\sin t} \end{aligned}$$

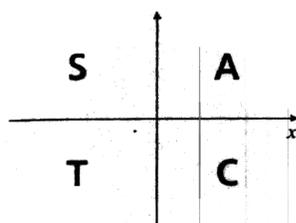


Figure P.81 The CAST rule

Observe that each of these functions is undefined (and its graph approaches vertical lines) at points where the function in the denominator of its defining fraction has value 0. Observe also that tangent, cotangent, and cosecant are odd functions and secant is an even function. Since  $|\sin x| \leq 1$  and  $|\cos x| \leq 1$  for all  $x$ ,  $|\csc x| \geq 1$  and  $|\sec x| \geq 1$  for all  $x$  where they are defined.

The three functions sine, cosine, and tangent are called the **primary trigonometric functions** while their reciprocals cosecant, secant, and cotangent are called the **secondary trigonometric functions**. Scientific calculators usually just implement the primary functions; you can use the reciprocal key to find values of the corresponding secondary functions. Figure P.81 shows a useful pattern called the “CAST rule” to help you remember where the primary functions are positive. All three are positive in the first quadrant, marked A. Of the three, only sine is positive in the second quadrant S, only tangent in the third quadrant T, and only cosine in the fourth quadrant C.

**Example 7** Find the sine and tangent of the angle  $\theta$  in  $[\pi, \frac{3\pi}{2}]$  for which

$$\cos \theta = -\frac{1}{3}.$$

**Solution** From the Pythagorean identity  $\sin^2 \theta + \cos^2 \theta = 1$  we get

$$\sin^2 \theta = 1 - \frac{1}{9} = \frac{8}{9}, \quad \text{so} \quad \sin \theta = \pm \sqrt{\frac{8}{9}} = \pm \frac{2\sqrt{2}}{3}.$$

The requirement that  $\theta$  should lie in  $[\pi, 3\pi/2]$  makes  $\theta$  a third quadrant angle. Its sine is therefore negative. We have

$$\sin \theta = -\frac{2\sqrt{2}}{3} \quad \text{and} \quad \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{-2\sqrt{2}/3}{-1/3} = 2\sqrt{2}.$$

Like their reciprocals cosine and sine, the functions secant and cosecant are periodic with period  $2\pi$ . Tangent and cotangent, however, have period  $\pi$  because

$$\tan(x + \pi) = \frac{\sin(x + \pi)}{\cos(x + \pi)} = \frac{\sin x \cos \pi + \cos x \sin \pi}{\cos x \cos \pi - \sin x \sin \pi} = \frac{-\sin x}{-\cos x} = \tan x.$$

Dividing the Pythagorean identity  $\sin^2 x + \cos^2 x = 1$  by  $\cos^2 x$  and  $\sin^2 x$ , respectively, leads to two useful alternative versions of that identity:

$$1 + \tan^2 x = \sec^2 x \quad \text{and} \quad 1 + \cot^2 x = \csc^2 x.$$

Addition formulas for tangent and cotangent can be obtained from those for sine and cosine. For example,

$$\tan(s + t) = \frac{\sin(s + t)}{\cos(s + t)} = \frac{\sin s \cos t + \cos s \sin t}{\cos s \cos t - \sin s \sin t}.$$

Now divide the numerator and denominator of the fraction on the right by  $\cos s \cos t$  to get

$$\tan(s + t) = \frac{\tan s + \tan t}{1 - \tan s \tan t}$$

Replacing  $t$  by  $-t$  leads to

$$\tan(s - t) = \frac{\tan s - \tan t}{1 + \tan s \tan t}$$

## Maple Calculations

Maple knows all six trigonometric functions and can calculate their values and manipulate them in other ways. It assumes the arguments of the trigonometric functions are in radians.

```
> evalf(sin(30)); evalf(sin(Pi/6));
      -0.9880316241
      .5000000000
```

Note that the constant Pi (with an upper case "P") is known to Maple. The `evalf()` function converts its argument to a number expressed as a floating point decimal with 10 significant digits. (This precision can be changed by defining a new value for the variable `Digits`.) Without it the sine of 30 radians would have been left unexpanded because it is not an integer.

```
> Digits := 20; evalf(100*Pi); sin(30);
      Digits := 20
      314.15926535897932385
      sin(30)
```

It is often useful to expand trigonometric functions of multiple angles to powers of sine and cosine, and vice versa.

```
> expand(sin(5*x));
      16 sin(x) cos(x)^4 - 12 sin(x) cos(x)^2 + sin(x)
> combine((cos(x))^5, trig);
      1/16 cos(5x) + 5/16 cos(3x) + 5/8 cos(x)
```

Other trigonometric functions can be converted to expressions involving sine and cosine.

```
> convert(tan(4*x)*(sec(4*x))^2, sincos); combine(% , trig);
      sin(4x)
      -----
      cos(4x)^3
      4 sin(4x)
      -----
      cos(12x) + 3 cos(4x)
```

The `%` in the last command referred to the result of the previous calculation.

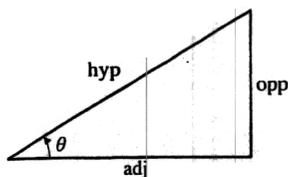


Figure P.82

## Trigonometry Review

The trigonometric functions are so called because they are often used to express the relationships between the sides and angles of a triangle. As we observed at the beginning of this section, if  $\theta$  is one of the acute angles in a right-angled triangle, we can refer to the three sides of the triangle as adj (side adjacent  $\theta$ ), opp (side opposite  $\theta$ ), and hyp (hypotenuse). The trigonometric functions of  $\theta$  can then be expressed as ratios of these sides, in particular:

$$\sin \theta = \frac{\text{opp}}{\text{hyp}}, \quad \cos \theta = \frac{\text{adj}}{\text{hyp}}, \quad \tan \theta = \frac{\text{opp}}{\text{adj}}$$

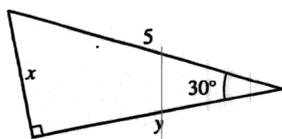


Figure P.83

**Example 8** Find the unknown sides  $x$  and  $y$  of the triangle in Figure P.83.

**Solution** Here  $x$  is the side opposite and  $y$  is the side adjacent the  $30^\circ$  angle. The hypotenuse of the triangle is 5 units. Thus

$$\frac{x}{5} = \sin 30^\circ = \frac{1}{2} \quad \text{and} \quad \frac{y}{5} = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\text{so } x = \frac{5}{2} \text{ units and } y = \frac{5\sqrt{3}}{2} \text{ units.}$$

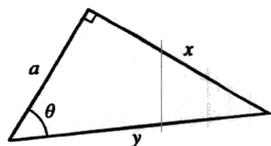


Figure P.84

**Example 9** For the triangle in Figure P.84, express sides  $x$  and  $y$  in terms of side  $a$  and angle  $\theta$ .

**Solution** The side  $x$  is opposite the angle  $\theta$  and  $y$  is the hypotenuse. The side adjacent  $\theta$  is  $a$ . Thus

$$\frac{x}{y} = \tan \theta \quad \text{and} \quad \frac{a}{y} = \cos \theta.$$

$$\text{Hence, } x = a \tan \theta \text{ and } y = \frac{a}{\cos \theta} = a \sec \theta$$

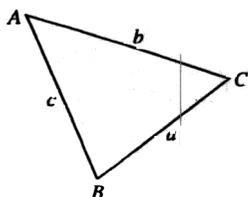


Figure P.85 In this triangle the sides are named to correspond to the opposite angles.

When dealing with general (not necessarily right-angled) triangles, it is often convenient to label the vertices with capital letters, which also denote the angles at those vertices, and refer to the sides opposite those vertices by the corresponding lower-case letters. See Figure P.85. Relationships among the sides  $a$ ,  $b$ , and  $c$  and opposite angles  $A$ ,  $B$ , and  $C$  of an arbitrary triangle  $ABC$  are given by the following formulas, called the **Sine Law** and the **Cosine Law**.

### THEOREM 2

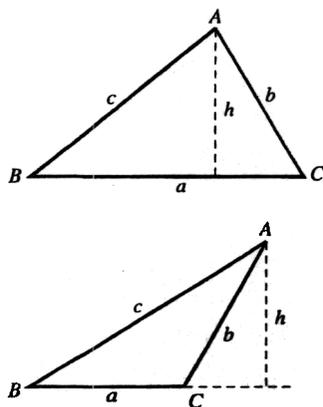
Sine Law:  $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Cosine Law:  $a^2 = b^2 + c^2 - 2bc \cos A$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

**PROOF** See Figure P.86. Let  $h$  be the length of the perpendicular from  $A$  to the side  $BC$ . From right-angled triangles (and using  $\sin(\pi - t) = \sin t$  if required), we get  $c \sin B = h = b \sin C$ . Thus  $(\sin B)/b = (\sin C)/c$ . By the symmetry of the formulas (or by dropping a perpendicular to another side), both fractions must be equal to  $(\sin A)/a$ , so the Sine Law is proved. For the Cosine Law, observe that



$$\begin{aligned}
 c^2 &= \begin{cases} h^2 + (a - b \cos C)^2 & \text{if } C \leq \frac{\pi}{2} \\ h^2 + (a + b \cos(\pi - C))^2 & \text{if } C > \frac{\pi}{2} \end{cases} \\
 &= h^2 + (a - b \cos C)^2 \quad (\text{since } \cos(\pi - C) = -\cos C) \\
 &= b^2 \sin^2 C + a^2 - 2ab \cos C + b^2 \cos^2 C \\
 &= a^2 + b^2 - 2ab \cos C.
 \end{aligned}$$

Figure P.86

The other versions of the Cosine Law can be proved in a similar way.

**Example 10** A triangle has sides  $a = 2$  and  $b = 3$  and angle  $C = 40^\circ$ . Find side  $c$  and the sine of angle  $B$ .

**Solution** From the third version of the Cosine Law:

$$c^2 = a^2 + b^2 - 2ab \cos C = 4 + 9 - 12 \cos 40^\circ \approx 13 - 12 \times 0.766 = 3.808.$$

Side  $c$  is about  $\sqrt{3.808} = 1.951$  units in length. Now using Sine Law we get

$$\sin B = b \frac{\sin C}{c} \approx 3 \times \frac{\sin 40^\circ}{1.951} \approx \frac{3 \times 0.6428}{1.951} \approx 0.988.$$

A triangle is uniquely determined by any one of the the following sets of data (which correspond to the known cases of congruency of triangles in classical geometry):

1. two sides and the angle contained between them (e.g., Example 10);
2. three sides, no one of which exceeds the sum of the other two in length;
3. two angles and one side; or
4. the hypotenuse and one other side of a right-angled triangle.

In such cases you can always find the unknown sides and angles by using the Pythagorean theorem or the Sine and Cosine Laws, and the fact that the sum of the three angles of a triangle is  $180^\circ$  (or  $\pi$  radians).

A triangle is not determined uniquely by two sides and a non-contained angle; there may exist no triangle, one right-angled triangle, or two triangles having such data.

**Example 11** In triangle  $ABC$ , angle  $B = 30^\circ$ ,  $b = 2$ , and  $c = 3$ . Find  $a$ .

**Solution** This is one of the ambiguous cases. By the Cosine Law

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$4 = a^2 + 9 - 6a(\sqrt{3}/2).$$

Therefore,  $a$  must satisfy the equation  $a^2 - 3\sqrt{3}a + 5 = 0$ . Solving this equation using the quadratic formula, we obtain

$$a = \frac{3\sqrt{3} \pm \sqrt{27 - 20}}{2}$$

$$\approx 1.275 \quad \text{or} \quad 3.921$$

There are two triangles with the given data, as shown in Figure P.87.

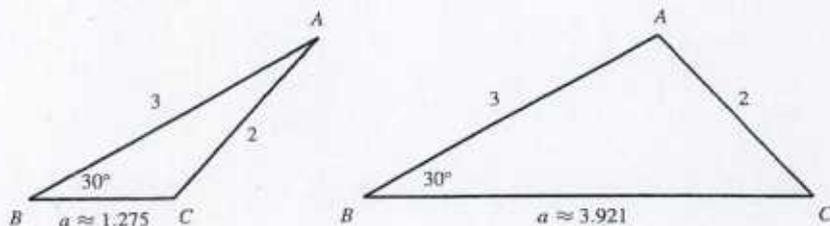


Figure P.87 Two triangles with  $b = 2$ ,  $c = 3$ ,  $B = 30^\circ$ .

## Exercises P.6

Find the values of the quantities in Exercises 1–6 using various formulas presented in this section. Do not use tables or a calculator.

1.  $\cos \frac{3\pi}{4}$       2.  $\tan -\frac{3\pi}{4}$       3.  $\sin \frac{2\pi}{3}$

4.  $\sin \frac{7\pi}{12}$       5.  $\cos \frac{5\pi}{12}$       6.  $\sin \frac{11\pi}{12}$

In Exercises 7–12, express the given quantity in terms of  $\sin x$  and  $\cos x$ .

7.  $\cos(\pi + x)$       8.  $\sin(2\pi - x)$       9.  $\sin\left(\frac{3\pi}{2} - x\right)$

10.  $\cos\left(\frac{3\pi}{2} + x\right)$       11.  $\tan x + \cot x$       12.  $\frac{\tan x - \cot x}{\tan x + \cot x}$

In Exercises 13–16, prove the given identities.

13.  $\cos^4 x - \sin^4 x = \cos(2x)$

14.  $\frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x} = \tan \frac{x}{2}$

15.  $\frac{1 - \cos x}{1 + \cos x} = \tan^2 \frac{x}{2}$

16.  $\frac{\cos x - \sin x}{\cos x + \sin x} = \sec 2x - \tan 2x$

17. Express  $\sin 3x$  in terms of  $\sin x$  and  $\cos x$ .

18. Express  $\cos 3x$  in terms of  $\sin x$  and  $\cos x$ .

In Exercises 19–22, sketch the graph of the given function. What is the period of the function?

19.  $f(x) = \cos 2x$       20.  $f(x) = \sin \frac{x}{2}$

21.  $f(x) = \sin \pi x$       22.  $f(x) = \cos \frac{\pi x}{2}$

23. Sketch the graph of  $y = 2 \cos\left(x - \frac{\pi}{3}\right)$ .

24. Sketch the graph of  $y = 1 + \sin\left(x + \frac{\pi}{4}\right)$ .

In Exercises 25–30, one of  $\sin \theta$ ,  $\cos \theta$ , and  $\tan \theta$  is given. Find the other two if  $\theta$  lies in the specified interval.

25.  $\sin \theta = \frac{3}{5}$ ,  $\theta$  in  $\left[\frac{\pi}{2}, \pi\right]$

26.  $\tan \theta = 2$ ,  $\theta$  in  $\left[0, \frac{\pi}{2}\right]$

27.  $\cos \theta = \frac{1}{3}$ ,  $\theta$  in  $\left[-\frac{\pi}{2}, 0\right]$

28.  $\cos \theta = -\frac{5}{13}$ ,  $\theta$  in  $\left[\frac{\pi}{2}, \pi\right]$

29.  $\sin \theta = \frac{-1}{2}$ ,  $\theta$  in  $\left[\pi, \frac{3\pi}{2}\right]$

30.  $\tan \theta = \frac{1}{2}$ ,  $\theta$  in  $\left[\pi, \frac{3\pi}{2}\right]$

**Trigonometry Review**

In Exercises 31–42,  $ABC$  is a triangle with a right angle at  $C$ . The sides opposite angles  $A$ ,  $B$ , and  $C$  are  $a$ ,  $b$ , and  $c$ , respectively. (See Figure P.88.)

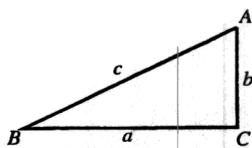


Figure P.88

31. Find  $a$  and  $b$  if  $c = 2$ ,  $B = \frac{\pi}{3}$ .
32. Find  $a$  and  $c$  if  $b = 2$ ,  $B = \frac{\pi}{3}$ .
33. Find  $b$  and  $c$  if  $a = 5$ ,  $B = \frac{\pi}{6}$ .
34. Express  $a$  in terms of  $A$  and  $c$ .
35. Express  $a$  in terms of  $A$  and  $b$ .
36. Express  $a$  in terms of  $B$  and  $c$ .
37. Express  $a$  in terms of  $B$  and  $b$ .
38. Express  $c$  in terms of  $A$  and  $a$ .
39. Express  $c$  in terms of  $A$  and  $b$ .
40. Express  $\sin A$  in terms of  $a$  and  $c$ .
41. Express  $\sin A$  in terms of  $b$  and  $c$ .
42. Express  $\sin A$  in terms of  $a$  and  $b$ .

In Exercises 43–52,  $ABC$  is an arbitrary triangle with sides  $a$ ,  $b$ , and  $c$ , opposite to angles  $A$ ,  $B$ , and  $C$ , respectively. (See Figure P.89.) Find the indicated quantities. Use tables or a scientific calculator if necessary.

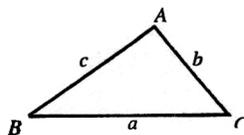


Figure P.89

43. Find  $\sin B$  if  $a = 4$ ,  $b = 3$ ,  $A = \frac{\pi}{4}$ .
44. Find  $\cos A$  if  $a = 2$ ,  $b = 2$ ,  $c = 3$ .
45. Find  $\sin B$  if  $a = 2$ ,  $b = 3$ ,  $c = 4$ .
46. Find  $c$  if  $a = 2$ ,  $b = 3$ ,  $C = \frac{\pi}{4}$ .
47. Find  $a$  if  $c = 3$ ,  $A = \frac{\pi}{4}$ ,  $B = \frac{\pi}{3}$ .
48. Find  $c$  if  $a = 2$ ,  $b = 3$ ,  $C = 35^\circ$ .
49. Find  $b$  if  $a = 4$ ,  $B = 40^\circ$ ,  $C = 70^\circ$ .
50. Find  $c$  if  $a = 1$ ,  $b = \sqrt{2}$ ,  $A = 30^\circ$ . (There are two possible answers.)
51. Two guy wires stretch from the top  $T$  of a vertical pole to points  $B$  and  $C$  on the ground, where  $C$  is 10 m closer to the base of the pole than is  $B$ . If wire  $BT$  makes an angle of  $35^\circ$  with the horizontal, and wire  $CT$  makes an angle of  $50^\circ$  with the horizontal, how high is the pole?
52. Observers at positions  $A$  and  $B$  2 km apart simultaneously measure the angle of elevation of a weather balloon to be  $40^\circ$  and  $70^\circ$ , respectively. If the balloon is directly above a point on the line segment between  $A$  and  $B$ , find the height of the balloon.
53. Show that the area of triangle  $ABC$  is given by  $(1/2)ab \sin C = (1/2)bc \sin A = (1/2)ca \sin B$ .
- \* 54. Show that the area of triangle  $ABC$  is given by  $\sqrt{s(s-a)(s-b)(s-c)}$ , where  $s = (a+b+c)/2$  is the semi-perimeter of the triangle.

\* This symbol is used throughout the book to indicate an exercise that is somewhat more difficult and/or theoretical than most exercises.