KOÇ UNIVERSITY

MATH 101 - FINITE MATHEMATICS

Midterm 2 November 28, 2015

Duration of Exam: 90 minutes

INSTRUCTIONS: You can NOT use calculators in the exam. No books, no notes, and no talking allowed. You must always **explain your answers** and **show your work** to receive **full credit**. Use the back of these pages if necessary. **Print (use CAPITAL LETTERS) and sign your name, and indicate your section below.**

Name: ______________________

Surname: SOLUTION

Signature: ______________________

Section (Check One):

Lecture 1: Mine Çağlar M-W (10:00) —
Lecture 2: Mine Çağlar M-W (13:00) —
Lecture 3: Ayberk Zeytin Tu-Th(13:00) —
Lecture 4: Ayberk Zeytin Tu-Th(16:00) —

PROBLEM	POINTS	SCORE
1	15	
2	25	
3	15	
4	20	
5	30	
TOTAL	105	

1. (15 points) Find all possible $\lambda \in \mathbb{R}$ for which the following matrix is not invertible.

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1-\lambda \end{pmatrix}$$

$$\xrightarrow{-R_3+R_1 \to R_1} B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1-\lambda \end{bmatrix} \Rightarrow \det A = \det B$$

$$\det A = \det B = B_{11} C_{11} = (1)(-1)^{1+1} \det \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix}}_{D}$$

$$= D_{12} (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 0 & 1-\lambda \end{vmatrix}$$

$$= (-1)[(1)(1-\lambda) - 0] = -(1-\lambda) = \lambda - 1$$

If $\lambda - 1 \neq 0$, that is, $\lambda \neq 1$, then $\det A \neq 0$, and hence it is invertible.

2. (a) (20 points) Using Gauss-Jordan elimination find the inverse of the matrix

$$B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 6 & 3 & 1 \end{pmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 6 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1/2 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 6 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-6R_1+R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2+R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1/2 & -1/2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-2R_3+R_2 \to R_2]{R_3+R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5/2 & -1/2 & 1 \\ 0 & 1 & 0 & 6 & 1 & -2 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right] \Rightarrow B^{-1} = \begin{bmatrix} -5/2 & -1/2 & 1 \\ 6 & 1 & -2 \\ -3 & 0 & 1 \end{bmatrix}$$

(b) (5 points) Solve the following system using the inverse of B you found in part (a).

$$\begin{aligned} 2x_1 + x_2 &= 4 \\ x_2 + 2x_3 &= -7 \\ 6x_1 + 3x_2 + x_3 &= 7 \end{aligned}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$BX = \begin{bmatrix} 4 \\ -7 \\ 7 \end{bmatrix}$$

$$\Rightarrow X = B^{-1}\begin{bmatrix} 4 \\ -7 \\ 7 \end{bmatrix} = \begin{bmatrix} -5/2 & -1/2 & 1 \\ 6 & 1 & -2 \\ -3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ -7 \\ 7 \end{bmatrix}$$

$$X = \begin{bmatrix} 1/2 \\ 3 \\ -5 \end{bmatrix}$$

$-\frac{5}{2}(4) + \frac{7}{2} + 7 = -10 + 7 + \frac{7}{2}$
$= -3 + \frac{7}{2} = \frac{1}{2}$

$24 - 7 - 14 = 3$

$-12 + 7 = -5$

3. (a) (5 points) Find the matrix A, if $A = CD + ED$, where

$$C = \begin{pmatrix} -1 & 1 & 4 & -2 \\ 0 & 1 & -3 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 \\ 2 & 1 \\ 1 & 3 \\ 0 & 1 \end{pmatrix}, \text{ and } E = \begin{pmatrix} 2 & 0 & 3 & -2 \\ 6 & 1 & -1 & 0 \end{pmatrix}.$$

$$A = (C+E)D = \left(\begin{bmatrix} -1 & 1 & 4 & -2 \\ 0 & 1 & -3 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 3 & -2 \\ 6 & 1 & -1 & 0 \end{bmatrix} \right) \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 7 & -4 \\ 6 & 2 & -4 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 18 \\ -6 & -11 \end{bmatrix}$$

$-1+2+7 = 8$

$1+21-4 = 18$

$-6+4-4 = -6$

$2-12-1 = -11$

(b) (10 points) Use Cramer's rule to solve AX=B, where $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $B = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$, and A is as given in part (a).

$$\begin{bmatrix} 8 & 17 \\ -6 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$$\det A = 8(-11) - (-6)(17) = -88 + 102 = 14$$

$$x_1 = \frac{1}{14} \begin{vmatrix} 5 & 17 \\ -1 & -11 \end{vmatrix} = \frac{1}{14}(-55+17) = -\frac{38}{14} = \frac{-19}{7}$$

$$x_2 = \frac{1}{14} \begin{vmatrix} 8 & 5 \\ -6 & -1 \end{vmatrix} = \frac{1}{14}(-8+30) = \frac{22}{14} = \frac{11}{7}$$

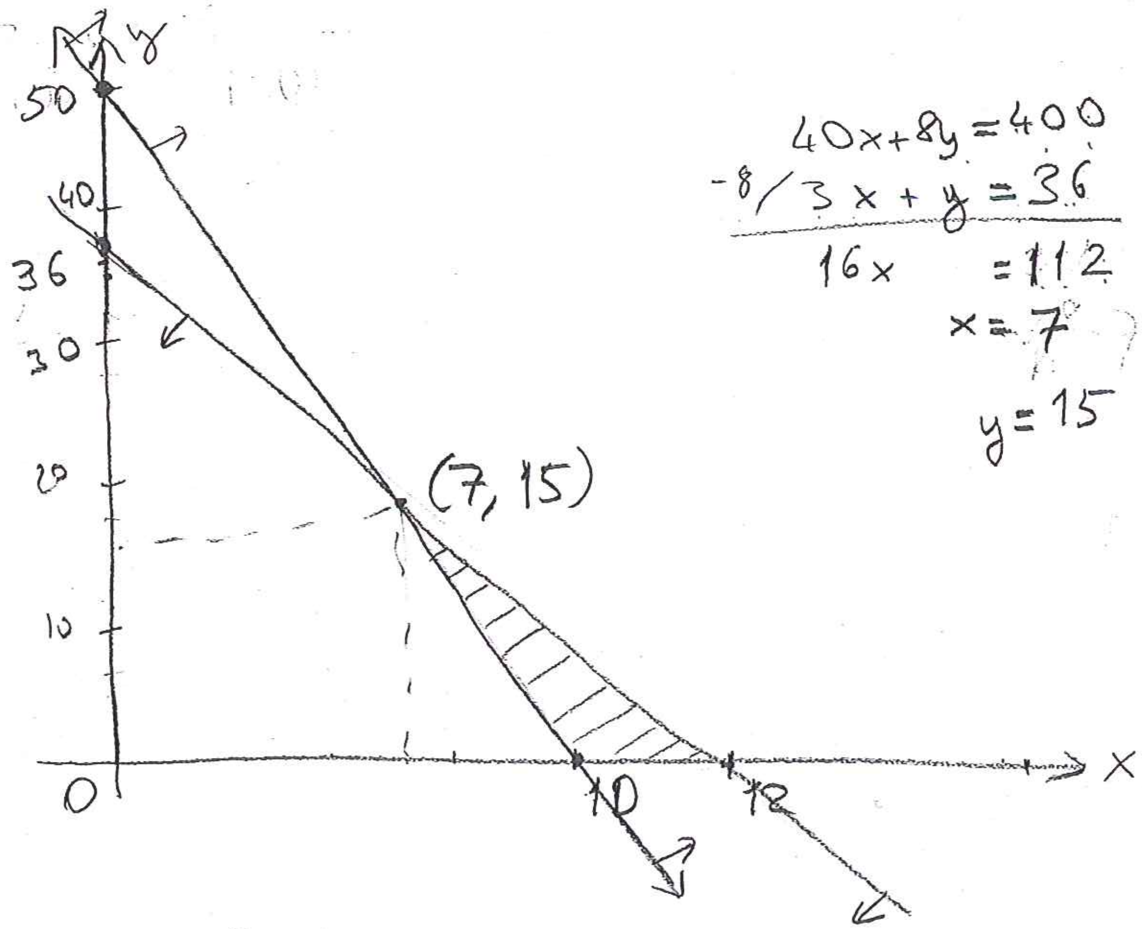
Let x: number of buses to rent
y: number of minibuses to rent

4. (20 points) A tourism agency will plan for a day trip. There are 400 customers who have applied for this trip. Each bus can transport 40 passengers, requires 3 tourist guides, and costs $1,200 to rent. Each minibus can transport 8 passengers, requires 1 guide, and costs $100 to rent. Only 36 tourist guides have been hired in the agency. How many vehicles of each type should the agency rent in order to minimize the transportation costs? What is the minimal transportation cost?

Minimize $C = 1200x + 100y$

Subject to $40x + 8y \geq 400$
$3x + y \leq 36$
$x, y \geq 0$

$3x + y = 36$
$\to (0,36), (12,0)$
$40x + 8y = 400$
$\to (0,50), (10,0)$



$$\begin{aligned} 40x + 8y &= 400 \\ -8/\ 3x + y &= 36 \\ \hline 16x &= 112 \\ x &= 7 \\ y &= 15 \end{aligned}$$

Corner Points

x	y	C
7	15	9900 * $\to$ Minimum cost
10	0	12,000
12	0	14,400

$1200(7) + 100(15) = 8400 + 1500$
$= 9900$

Optimal sol^n: 7 buses
15 minibuses

5. (a) (20 points) Maximize $P = 2x_1 + 3x_2 + 2x_3$ subject to the following constraints

$$2x_1 + x_2 + x_3 \overset{+s_1 = 4}{\leq} 4$$
$$x_1 + 2x_2 + x_3 \overset{+s_2 = 7}{\leq} 7$$
$$x_1 \geq 0,\ x_2 \geq 0 \quad x_3 \geq 0$$

$$\begin{array}{c} \\ s_1 \\ s_2 \\ P \end{array}\begin{array}{c} x_1 \quad x_2 \quad x_3 \quad s_1 \quad s_2 \quad P \\ \left[\begin{array}{cccccc|c} 2 & 1 & 1 & 1 & 0 & 0 & 4 \\ 1 & 2 & 1 & 0 & 1 & 0 & 7 \\ \hline -2 & -3 & -2 & 0 & 0 & 1 & 0 \end{array}\right]\end{array}$$

$\frac{1}{2}R_2 \to R_2$

$$\left[\begin{array}{cccccc|c} 2 & 1 & 1 & 1 & 0 & 0 & 4 \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{7}{2} \\ \hline -2 & -3 & -2 & 0 & 0 & 1 & 0 \end{array}\right]$$

$-R_2 + R_1 \to R_1$

$3R_2 + R_3 \to R_3$

$$\begin{array}{c} s_1 \\ x_2 \\ P \end{array}\left[\begin{array}{cccccc|c} \frac{3}{2} & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{7}{2} \\ \hline -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{3}{2} & 1 & \frac{21}{2} \end{array}\right]$$

$2R_1 \to R_1$

$$\begin{array}{c} x_1 \quad x_2 \quad x_3 \quad s_1 \quad s_2 \quad P \\ \left[\begin{array}{cccccc|c} 3 & 0 & 1 & 2 & -1 & 0 & 1 \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{7}{2} \\ \hline -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{3}{2} & 1 & \frac{21}{2} \end{array}\right]\end{array}$$

$-\frac{1}{2}R_1 + R_2 \to R_2$

$\frac{R_1}{2} + R_3 \to R_3$

$$\begin{array}{c} x_3 \\ x_2 \\ P \end{array}\left[\begin{array}{cccccc|c} 3 & 0 & 1 & 2 & -1 & 0 & 1 \\ -1 & 1 & 0 & -1 & 1 & 0 & 3 \\ \hline 1 & 0 & 0 & 1 & 1 & 1 & 11 \end{array}\right]$$

Optimal Solution:

$x_1 = 0,\ x_2 = 3,\ x_3 = 1$

$P = 11$

(b) (10 points) Complete the missing rows in the following table of basic solutions of the linear programming problem given in part (a).

x_1	x_2	x_3	s_1	s_2	P
0	0	0	4	7	0
0	0	4	0	3	8
0	0	7	-3	0	X
0	4	0	0	-1	X
0	7/2	0	1/2	0	21/2
0	3	1	0	0	11
2	0	0	0	5	4
7	0	0	-10	0	X
-3	0	10	0	0	X
1/3	10/3	0	0	0	32/3