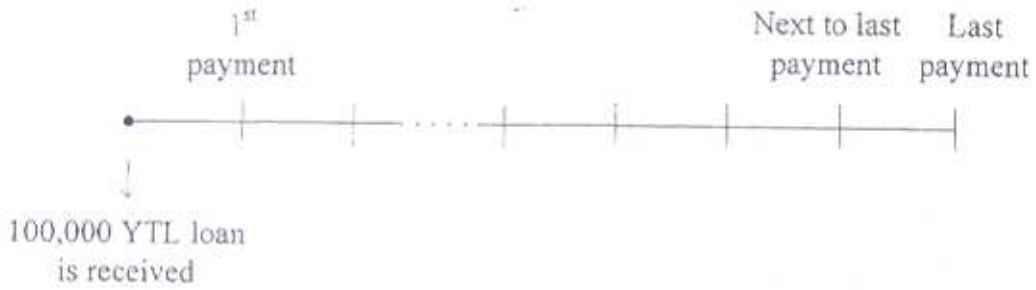
# MATH 101 Fall 2005 MT2 Solutions

A list of formulas: $I = Prt$; $A = P(1+rt)$

$A = P(1+i)^n$; $APY = (1+r/m)^m - 1$; $APY$: effective rate

$FV = PMT\dfrac{(1+i)^n - 1}{i}$; $PV = PMT\dfrac{1-(1+i)^{-n}}{i}$; $i = \dfrac{r}{m}$; $n = mt$

1. A family buys a house with 100,000 YTL loan to be paid off in ten years by equal monthly payments. The loan company charges 1% monthly interest on the unpaid balance.




a) (5 points) What is the monthly payment?

$PV = 100{,}000$ YTL

$\frac{r}{m} = 0.01$

$n = 120$

$$PV = PMT\,\frac{1-(1+\frac{r}{m})^{-n}}{\frac{r}{m}}$$

$$PMT = PV\cdot\frac{r}{m}\,\frac{1}{1-(1+\frac{r}{m})^{-n}} = 1434.71 \text{ YTL}$$

b) (10 points) How much interest does the family pay in the first year?

After 1 year, the remaining debt is

$$PV = \underbrace{PMT}_{1434.71\text{YTL}}\ \frac{1-(1+0.01)^{-108}}{0.01} = 94{,}484.83 \text{ YTL}$$

← present value of the debt after the first year.

The amount paid during the first year $= 1434.71 \times 12 = 17{,}216.52$ YTL

The amount that goes to the reduction of debt $= 100{,}000 - 94{,}484.83 = 5{,}515.17$ YTL

The amount that goes to the interest $= 17{,}216.52 - 5{,}515.17 = \boxed{11{,}701.35 \text{ YTL}}$

c) (5 points) How much interest does the family pay totally?

The amount paid in 10 years $= 120 \times 1{,}434.71 = 172{,}165.2$ YTL

Total interest paid $= 172{,}165.2 - 100{,}000 = \underline{\underline{72{,}165.2}}$ YTL

d) (5 points) How much does the family owe just after the next to last payment?

$PMT\left(1+\frac{r}{m}\right)^{-1} = \underline{\underline{1429.5 \text{ YTL}}}$

$PV = PMT\,\dfrac{1-(1+0.01)^{-1}}{0.01} = PMT\,(1+0.01)^{-1}$

present value of the debt after paying next-to-last payment, as it remains 1 month to pay the last payment.

**2.** a) (12 points) Solve the following system using Gauss-Jordan method.

$$\begin{aligned} x_1 + x_2 - x_4 &= -3 \\ 2x_1 - 2x_3 + 4x_4 &= 2 \\ 3x_1 - x_2 + 3x_3 &= -1 \end{aligned}$$

Please use the table, or otherwise indicate what you do at each step.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & -3 \\ 2 & 0 & -2 & 4 & 2 \\ 3 & -1 & 3 & 0 & -1 \end{array}\right] \rightarrow$$

| | | |
|---|---|---|
| Exchanged | Row ___ | with Row ___ |
| Added | -2 times Row 1 | to Row 2 |
| Multiplied | Row ___ | by ___ |

to get →

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & -3 \\ 0 & -2 & -2 & 6 & 8 \\ 3 & -1 & 3 & 0 & -1 \end{array}\right] \rightarrow$$

| | | |
|---|---|---|
| Exchanged | Row ___ | with Row ___ |
| Added | -3 times Row 1 | to Row 3 |
| Multiplied | Row ___ | by ___ |

to get →

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & -3 \\ 0 & -2 & -2 & 6 & 8 \\ 0 & -4 & 3 & 3 & 8 \end{array}\right] \rightarrow$$

| | | |
|---|---|---|
| Exchanged | Row ___ | with Row ___ |
| Added | ___ times Row ___ | to Row ___ |
| Multiplied | Row 2 | by $-\frac{1}{2}$ |

to get →

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & -3 & -4 \\ 0 & -4 & 3 & 3 & 8 \end{array}\right] \rightarrow$$

| | | |
|---|---|---|
| Exchanged | Row ___ | with Row ___ |
| Added | -1 times Row 2 | to Row 1 |
| Multiplied | Row ___ | by ___ |

to get →

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & 1 & -3 & -4 \\ 0 & -4 & 3 & 3 & 8 \end{array}\right] \rightarrow$$

| | | |
|---|---|---|
| Exchanged | Row ___ | with Row ___ |
| Added | 4 times Row 2 | to Row 3 |
| Multiplied | Row ___ | by ___ |

to get →

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & 1 & -3 & -4 \\ 0 & 0 & 7 & -9 & -8 \end{array}\right] \rightarrow$$

| | | |
|---|---|---|
| Exchanged | Row ___ | with Row ___ |
| Added | ___ times Row ___ | to Row ___ |
| Multiplied | Row 3 | by $\frac{1}{7}$ |

to get →

**2.** continued

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & 1 & -3 & -4 \\ 0 & 0 & 1 & -9/7 & -8/7 \end{array}\right] \rightarrow$$

| | | |
|---|---|---|
| Exchanged | Row ___ | with Row ___ |
| Added | −1 times Row 3 | to Row 2 |
| Multiplied | Row ___ | by ___ |

to get →

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & 0 & -12/7 & -20/7 \\ 0 & 0 & 1 & -9/7 & -8/7 \end{array}\right] \rightarrow$$

| | | |
|---|---|---|
| Exchanged | Row ___ | with Row ___ |
| Added | 1 times Row 3 | to Row 1 |
| Multiplied | Row ___ | by ___ |

to get →

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 5/7 & -1/7 \\ 0 & 1 & 0 & -12/7 & -20/7 \\ 0 & 0 & 1 & -9/7 & -8/7 \end{array}\right]$$

$$x_1 + 5/7\, x_4 = -1/7$$
$$x_2 - 12/7\, x_4 = -\frac{20}{7}$$
$$x_3 - 9/7\, x_4 = -8/7$$

Let $x_4 = t$, $t \in \mathbb{R}$

$$x_1 = -1/7 - 5/7\, t$$
$$x_2 = -\frac{20}{7} + \frac{12}{7} t$$
$$x_3 = -\frac{8}{7} + \frac{9}{7} t$$

For the systems in b) and c)

- write the system of equations each corresponds to
- find the solution(s) if it exists
- indicate if the system is consistent or inconsistent

b) (4 points) $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & -2 \end{array}\right]$

$x_1 + x_3 = 4$

$x_2 = -2$

consistent

$\Rightarrow$ Let $x_3 = t$, $t \in \mathbb{R}$

$\Rightarrow x_1 = 4 - t$

Solutions: $\{(4-t, -2, t) : t \in \mathbb{R}\}$

c) (4 points) $\left[\begin{array}{cccc|c} 2 & 1 & 0 & 5 & 6 \\ 0 & 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$2x_1 + x_2 + 5x_4 = 6$

$x_3 - 3x_4 = 4$

$0 = 1 \Rightarrow$ inconsistent, no solution.

$0 = 0$

Q.3 (Answer sheet)

(a)
$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 11 \\ 9 \\ 8 \end{bmatrix}}_{B}$$

(b)
$$[A:I] = \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - R_3 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - R_3 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - R_1 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 & 0 & 2 \end{array}\right] \xrightarrow{R_3 - R_2 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 3 \end{array}\right] = [I : A^{-1}] .$$

$$\boxed{A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 3 \end{bmatrix}}$$

(Note that A is diagonally symmetric, therefore so is $A^{-1}$)

(c)
$$X = A^{-1}B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 3 \end{bmatrix}\begin{bmatrix} 11 \\ 9 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

$x_1 = 3$ tons
$x_2 = 1$ ton
$x_3 = 4$ tons

(d)
$$B_{new} = \begin{bmatrix} 11 \\ 13 \\ 8 \end{bmatrix} \Rightarrow X_{new} = A^{-1}B_{new} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 3 \end{bmatrix}\begin{bmatrix} 11 \\ 13 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}$$

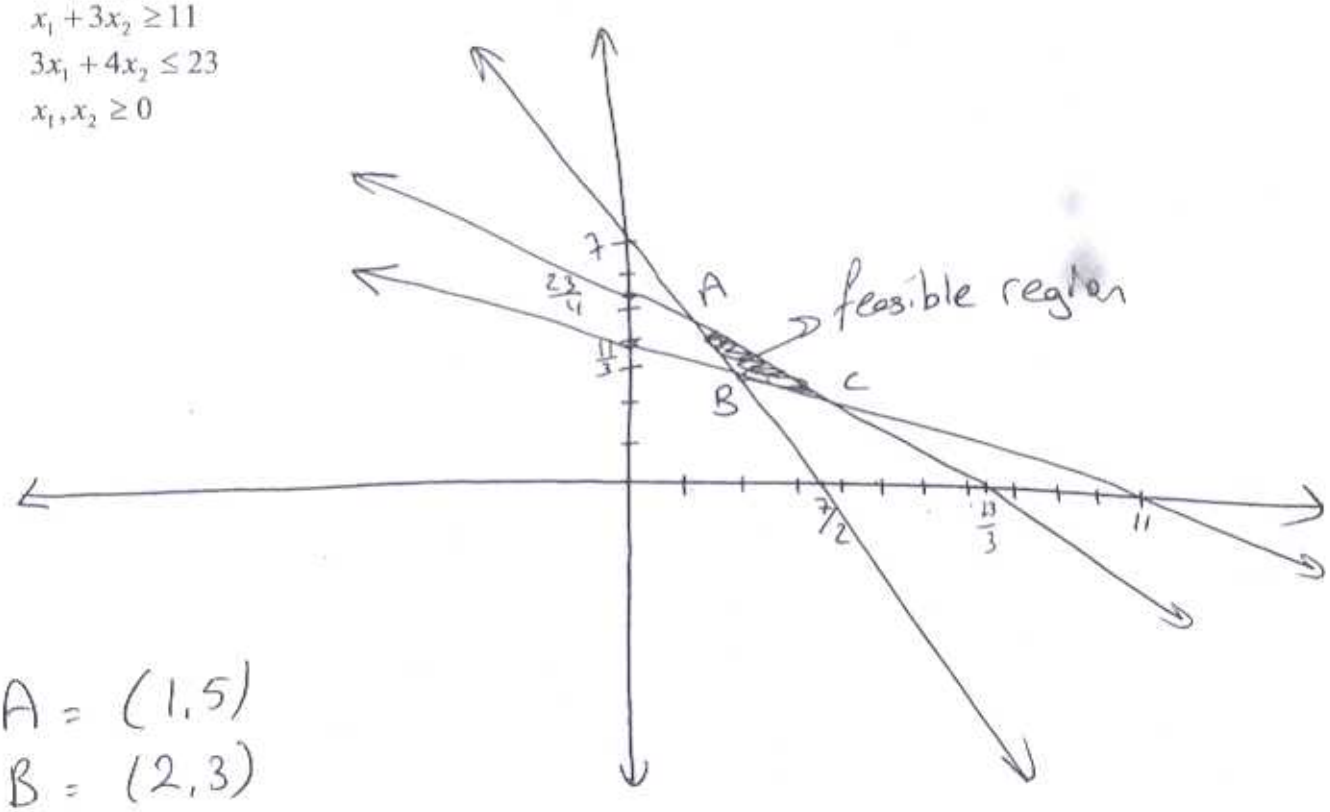
$x_1 = 3$ tons
$x_2 = 5$ tons
$x_3 =$ none

4) Minimize and maximize

$z = 10x_1 + 20x_2$

subject to

$2x_1 + x_2 \geq 7$
$x_1 + 3x_2 \geq 11$
$3x_1 + 4x_2 \leq 23$
$x_1, x_2 \geq 0$




A = (1, 5)
B = (2, 3)
C = (5, 2)

| Corner Points | $10x_1 + 20x_2$ |
|---|---|
| (1, 5) | 110 → max |
| (2, 3) | 80 → min |
| (5, 2) | 90 |

**5.** (10 points) Ebru Hanım has a workshop making sewn gifts for children. This month she is making 4 kinds of gifts: a dog, a cat, a camel, and a turtle.

She uses 4 different fabrics (cloth materials).

The dog takes 25 cm of fabric A, 10 cm of material B, none of fabric C and 15 cm of fabric D.

The cat takes 5 cm of fabric A, 15 cm of fabric B, 50 cm of fabric C and 2 cm of fabric D.

The camel takes 20 cm of fabric A, 20 cm of fabric B, 10 cm of fabric C and 10 cm of fabric D.

The turtle takes 8 cm of fabric A, 10 cm of fabric B, 20 cm of fabric C and 10 cm of fabric D.

Ebru Hanım has 3 metres of fabric A, 4 metres of fabric B, 2 metres of fabric C, and 5 metres of fabric D.

The profit on a dog is 8 YTL. The profit on a cat is 3 YTL. The profit on a camel is 6 YTL, and the profit on a turtle is 4 YTL.

How many of each gift should Ebru Hanım produce to make as large a profit as possible?

Write the decision variables, appropriate equation(s) and inequalities so that this problem can be solved.

DO NOT SOLVE THE PROBLEM after you have written the equation(s) and inequalities.

**Solution.** $x_1 = \#dogs$, $x_2 = \#cats$, $x_3 = \#camels$, $x_4 = \#turtles$, and $P=$ profit.

Subject to the constraints

$$\begin{aligned} 25x_1 + 5x_2 + 20x_3 + 8x_4 &\leq 300, \\ 10x_1 + 15x_2 + 20x_3 + 10x_4 &\leq 400, \\ 50x_2 + 10x_3 + 20x_4 &\leq 200, \\ 15x_1 + 2x_2 + 10x_3 + 10x_4 &\leq 500, \\ x_1, x_2, x_3, x_4 &\geq 0, \end{aligned}$$

maximize the objective function:

$$8x_1 + 3x_2 + 6x_3 + 4x_4 = P.$$