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December 16, 2005

Fall 2005, Math 101 FINAL EXAM

Show all your work to get full credit

Closed book & notes, calculator allowed, 2 hours and 15 minutes
ABSOLUTELY NO QUESTIONS WILL BE ANSWERED ABOUT THE EXAM, BY ANYONE, DURING THE EXAM.

Instructions: There are six questions in this exam. Please inspect the exam and make sure you have all 7 pages of questions. Do all your work on these pages. If you use the back of a page, make sure to indicate that.

Remember: *You must show your work to get proper credit.*

Academic Honesty Code: Koç University Academic Honesty Code stipulates that "copying from others or providing answers or information, written or oral, to others is cheating." By taking this exam, you are assuming full responsibility for observing the Academic Honesty Code.

NAME: ______________________________

Please check one:

İrşadi Aksun ☐
Colin Graham ☐
Alkan Kabakçıoğlu ☐
Şule Yazıcı 11:00-12:15 ☐
14:00-15:15 ☐
Mine Çağlar ☐

1	/15
2	/20
3	/15
4	/20
5	/20
6	/10
Total:	/100

A list of formulas: $I = Prt$; $A = P(1+rt)$

$A = P(1+i)^n$; $APY = (1+r/m)^m - 1$; APY: effective rate

$FV = PMT\dfrac{(1+i)^n - 1}{i}$; $PV = PMT\dfrac{1-(1+i)^{-n}}{i}$; $i = \dfrac{r}{m}$; $n = mt$

1. A bank offers two investment plans: i) 12% interest compounded monthly; ii) 12.6% interest compounded semi-annually. A family plans to invest 10,000 YTL at the end of every year into one of these plans for ten years to accumulate some money to buy a house.

a) (4 points) Which plan returns more interest (comparing their annual percentage yield also known as effective rate)?

$$APY_1 = \left(1 + \frac{0.12}{12}\right)^{12} - 1 = 0.126$$

$$APY_2 = \left(1 + \frac{0.126}{2}\right)^{2} - 1 = 0.1300$$

Second plan returns more interest.

b) (5 points) How much money would the family have in their account at the end of ten years, if they have invested into the plan that returns more interest?

$$FV = 10{,}000\,\frac{(1+0.13)^{10} - 1}{0.13}$$

$$= 184{,}197.50 \text{ YTL}$$

c) (6 points) The house they want to buy is worth 200,000 YTL. If the accumulated money in part b) is not enough to buy the house, the family plans to get a loan from the bank for the remaining part. The bank requires 1,000 YTL equal monthly payments with 15% interest on the unpaid balance. How long would it take for the family to pay back the loan?

They need to get $200{,}000 - 184{,}197.50 = 15{,}802.50$ YTL from the bank.

$$15{,}802.50 = 1000\,\frac{1-\left(1+\frac{0.15}{12}\right)^{-n}}{\frac{0.15}{12}}$$

$$\Rightarrow \left(1+\frac{0.15}{12}\right)^{-n} = \underbrace{1 - \frac{(15{,}802.50)}{1000}\,\frac{0.15}{12}}$$

$$-n\ln(1.0125) = \ln(0.80246875)$$

$$\Rightarrow n = 17.714\ldots \Rightarrow \boxed{n = 18 \text{ months}}$$

2. Answer the following.

Find the numerical value of x in parts a) through c)

a) (4 points) $\log_x e^{-2} = 2$

$$e^{-2} = x^2$$

$$-2 = \ln x^2 \qquad -2 = 2\ln x \qquad -1 = \ln x \quad \Rightarrow \quad x = e^{-1}$$

b) (5 points) $2^{1+\log_3 x} = 6^{\log_3 x}$

$$2^{1+\log_3 x} = 2^{\log_3 x} \cdot 3^{\log_3 x} \qquad (\text{since } 6 = 2 \cdot 3)$$

$$2 \cdot \cancel{2^{\log_3 x}} = \cancel{2^{\log_3 x}} \cdot x$$

$$\Rightarrow \quad x = 2$$

c) (5 points) $\operatorname{arcsec}(4\cos x) = x \quad \Longrightarrow \quad x \in [0, \pi]$ (range of arcsec)

$$\sec x = 4\cos x$$

$$\frac{1}{\cos x} = 4\cos x \quad \Rightarrow \quad \cos^2 x = \frac{1}{4}$$

$$\Rightarrow \quad \cos x = \mp\sqrt{\frac{1}{4}} \quad \Rightarrow \quad x = \pi/3 \text{ or } x = \frac{2\pi}{3}.$$

d) (6 points) Find the function $f(x)$ if

$$f(x)\sec^2 x - \tan^2 x = 2$$

and identify its range.

$$f(x) = \frac{2+\tan^2 x}{\sec^2 x} = \frac{2 + \frac{\sin^2 x}{\cos^2 x}}{\frac{1}{\cos^2 x}} = 2\cos^2 x + \sin^2 x$$

$$= 1 + \cos^2 x$$

$$-1 \le \cos x \le 1 \quad \Rightarrow \quad 0 \le \cos^2 x \le 1$$

$$\Rightarrow \text{Range of } f: \quad [1, 2]$$

3. a) (10 points) Minimize and maximize

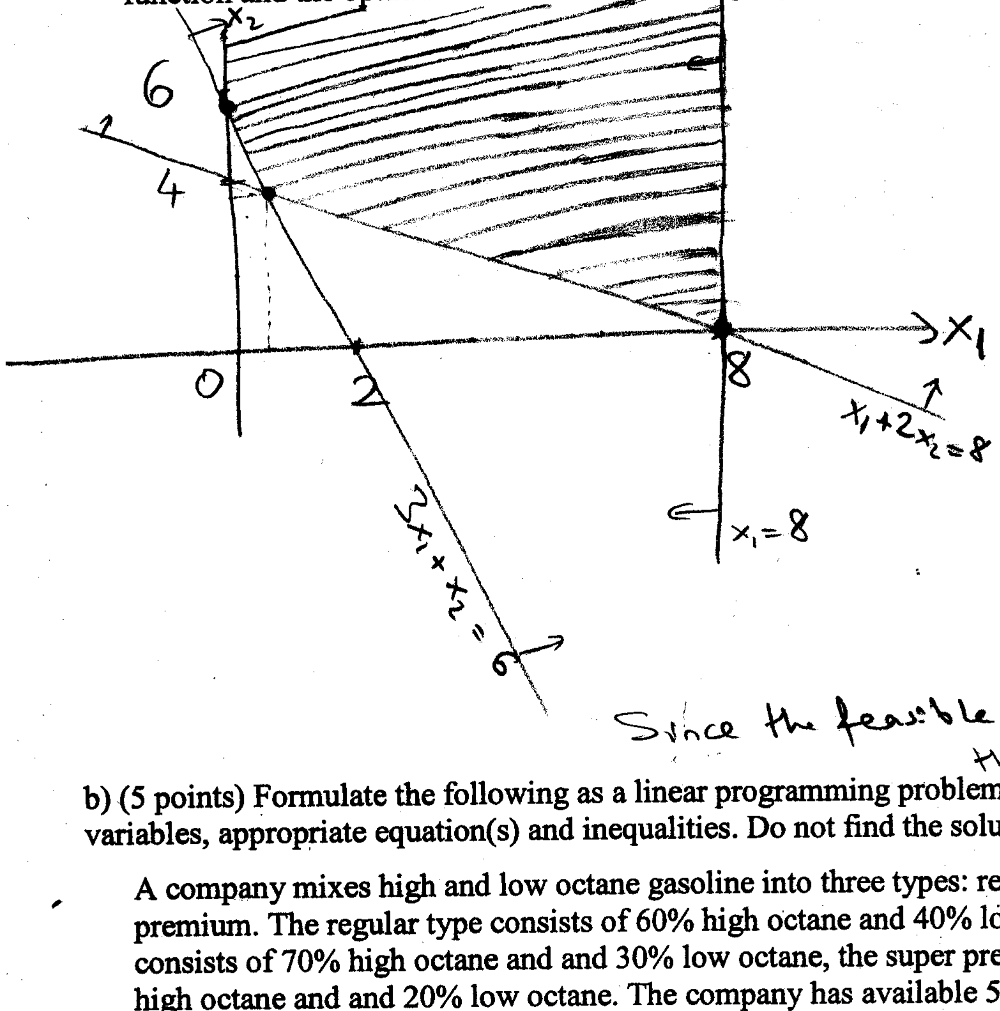
$$z = 3x_1 + 2x_2$$

subject to

$$x_1 + 2x_2 \geq 8$$
$$3x_1 + x_2 \geq 6$$
$$x_1 \leq 8$$
$$x_1, x_2 \geq 0$$

$$\left.\begin{array}{r} x_1 + 2x_2 = 8 \\ -2/3x_1 + x_2 = 6 \end{array}\right\} \Rightarrow -5x_1 = -4 \Rightarrow x_1 = \frac{4}{5} \Rightarrow x_2 = 6 - \frac{12}{5} = \frac{18}{5}$$

by sketching the graph of the feasible region. Indicate the optimal value of the objective function and the optimal solution to this linear programming problem explicitly.



Corner Points	z
$(0, 6)$	12
$(8, 0)$	24
$(4/5, 18/5)$	$\frac{48}{5}$

Minimum occurs at $x_1 = 4/5$, $x_2 = 18/5$ and is equal to $48/5$

Since the feasible region is unbounded, there is no maximum.

b) (5 points) Formulate the following as a linear programming problem; that is, write the decision variables, appropriate equation(s) and inequalities. Do not find the solution.

A company mixes high and low octane gasoline into three types: regular, premium and super premium. The regular type consists of 60% high octane and 40% low octane, the premium consists of 70% high octane and and 30% low octane, the super premium consists of 80% high octane and and 20% low octane. The company has available 560,000 litres of high octane and 480,000 litres of low octane, but is allowed to mix at most 900,000 litres of gasoline in total by government regulations. Regular gasoline sells for 2.6 YTL per litre, premium sells for 2.8YTL per litre and super premium sells for 3.1 YTL per litre. The company wants to know how many litres of each type it should mix in order to maximize its revenue.

Let x_1: litres of regular gasoline, x_2: litres of premium gasoline, x_3: litres of super premium gasoline

Maximize $2.6x_1 + 2.8x_2 + 3.1x_3$

Subject to

$$0.60x_1 + 0.70x_2 + 0.80x_3 \leq 560{,}000$$
$$0.40x_1 + 0.30x_2 + 0.20x_3 \leq 480{,}000$$
$$x_1 + x_2 + x_3 \leq 900{,}000$$
$$x_1 \geq 0, \quad x_2 \geq 0 \quad x_3 \geq 0$$

4. a) For the matrices A, B, C below, do each computation indicated, or explain why it cannot be done.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 4 \end{bmatrix}$$

i) (2 points) 2A+B

$$2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$$

ii) (2 points) AC

A is a 2×2 matrice
C is a 3×2 matrice
$\Rightarrow$ A and C cannot be multiplied.

iii) (2 points) CA

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 4 \end{bmatrix}$$

b) (4 points) Give all solutions (if any) for

$$x + y + z = 1$$
$$x + 2y + z = 2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right]$$ reduced $\Rightarrow$ $x_1 + x_3 = 0$, $x_2 = 1$

$x_3 = t \Rightarrow x_1 = -t$ $\Rightarrow \{(-t, 1, t) : t \in \mathbb{R}\}$ is the set of all solutions.

4. continued

c) (6 points) Find the inverse of $\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}}_{A}$. If you run out of space, please use the back of the previous page.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1+R_3\to R_3]{-R_1+R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[2R_2+R_3\to R_3]{-2R_2+R_1\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -2 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -3 & 2 & 1 \end{array}\right] \xrightarrow[-R_3+R_2\to R_2]{-R_3+R_1\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -4 & -1 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & -3 & 2 & 1 \end{array}\right]$$

$\Rightarrow$ Inverse is $\begin{bmatrix} 6 & -4 & -1 \\ 2 & -1 & -1 \\ -3 & 2 & 1 \end{bmatrix}$

Check:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 6 & -4 & -1 \\ 2 & -1 & -1 \\ -3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

d) (4 points) Solve the following sets of linear equations. Hint: Your answer to the previous part may be useful.

i) (2 points)
$$\begin{aligned} x+2y+3z&=1 \\ x+3y+4z&=0 \\ x\qquad+2z&=0 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 6 & -4 & -1 \\ 2 & -1 & -1 \\ -3 & 2 & 1 \end{bmatrix}}_{A^{-1}}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ -3 \end{bmatrix}$$

A is as in c)

ii) (2 points)
$$\begin{aligned} x+2y+3z&=0 \\ x+3y+4z&=0 \\ x\qquad+2z&=2 \end{aligned}$$

$$\begin{bmatrix} 6 & -4 & -1 \\ 2 & -1 & -1 \\ -3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}$$

5. Evaluate the limits in a) through c). Specify infinite limits and if the limit does not exist give the reason.

a) (4 points) $\lim_{x\to 0}\frac{1-\cos^2 x}{x} = \lim_{x\to 0}\frac{\sin^2 x}{x} = \lim_{x\to 0}\underbrace{\frac{\sin x}{x}}_{\downarrow\, 1}\cdot\underbrace{\sin x}_{\downarrow\, 0} = 0.$

b) (4 points) $\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1} = \lim_{x\to 1}\frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)} = \lim_{x\to 1}\frac{(x-1)(\sqrt{x}+1)}{x-1}$

$= \lim_{x\to 1}(\sqrt{x}+1) = 2$

c) (4 points) $\lim_{x\to 1}\frac{1}{x^2-1} = \lim_{x\to 1}\frac{1}{(x-1)(x+1)}$

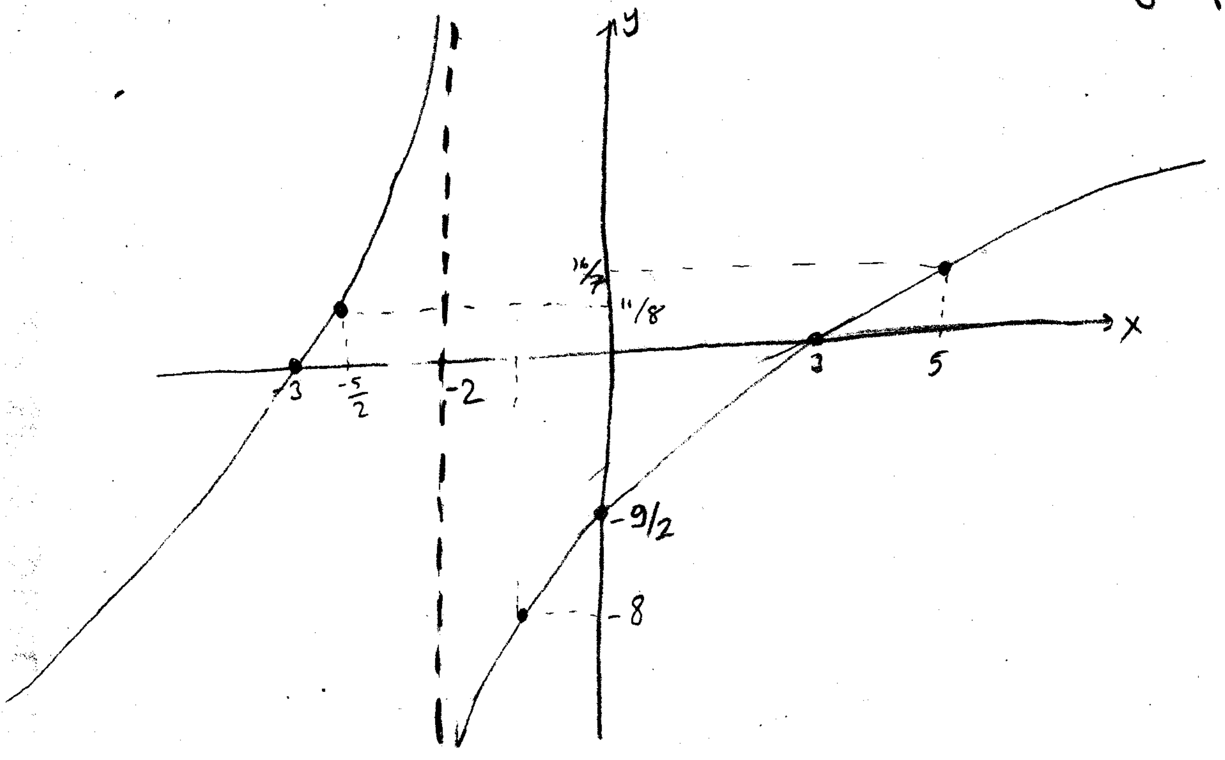
$\lim_{x\to 1^-}\frac{1}{\underbrace{(x-1)}_{<0}\underbrace{(x+1)}_{>0}} = -\infty \qquad \lim_{x\to 1^+}\frac{1}{\underbrace{(x-1)}_{>0}\underbrace{(x+1)}_{>0}} = \infty \Rightarrow$ limit does not exist.

d) (8 points) Find the vertical, horizontal and oblique asymptotes of $f(x)=\frac{x^2-9}{x+2}$ if they exist. Sketch the graph of f(x) by using these asymptotes.

No horizontal asymptote; $\lim_{x\to\infty}\frac{x^2-9}{x+2} = \infty$

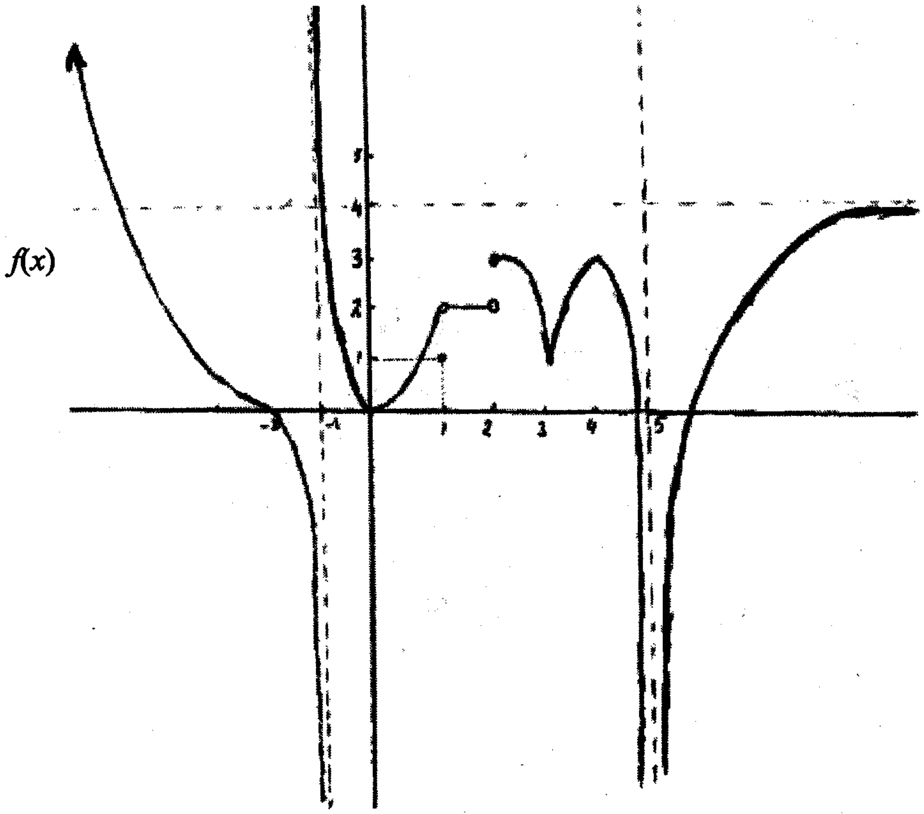
$\lim_{x\to-\infty}\frac{x^2-9}{x+2} = -\infty$

$x+2=0 \Rightarrow x=-2$ is the vertical asymptote.



x	y
-3	0
3	0
0	-9/2
-1	-8
5	16/7
-5/2	11/8

6. Answer the questions using the graph of $f(x)$ given below. Specify the infinite limits.



a) (1 point) $\lim_{x \to 1} f(x) =$ 2

b) (1 point) $f(1) =$ 1

c) (1 point) $\lim_{x \to -\infty} f(x) =$ ∞

d) (1 point) $\lim_{x \to 5} f(x) =$ $-\infty$

e) (1 point) $\lim_{x \to 2^+} f(x) =$ 3

f) (5 points) Find the points where f is discontinuous and explain why.

At points $\underbrace{x=-1}$, $\underbrace{x=1}$, $\underbrace{x=2}$, $\underbrace{x=5}$

$x=-1$	$x=1$	$x=2$	$x=5$
$\lim_{x \to -1^-} f(x) = -\infty$ $\lim_{x \to -1^+} f(x) = +\infty$ no limit	$\lim_{x \to 1} f(x) = 2$ but $f(1) = 1$ $1 \neq 2$	$\lim_{x \to 2^+} f(x) = 3$ $\neq \lim_{x \to 2^-} f(x) = 2$ limit does not exist	$\lim_{x \to 5} f(x) = -\infty$ limit does not exist