1. (a) (14 points) Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 5 & -2 & 0 & 1 \end{array}\right]$$

$(-2)R_1 + R_3 \to R_3$

$(-2)R_2 + R_3 \to R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & 1 \\ 0 & 1 & 0 & 4 & 5 & -2 \\ 0 & 0 & 1 & -2 & -2 & 1 \end{array}\right]$$

$(-2)R_3 + R_2 \to R_2$

$R_3 + R_1 \to R_1$

$$A^{-1} = \begin{bmatrix} -1 & -2 & 1 \\ 4 & 5 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

(b) (6 points) Using the inverse of A solve

$$\begin{cases} x_1 - x_3 &= 3 \\ x_2 + 2x_3 &= -2 \\ 2x_1 + 2x_2 + 3x_3 &= 1 \end{cases}$$

$A \cdot x = C$

$\underbrace{A^{-1} \cdot A}_{I} \cdot x = A^{-1} \cdot C$

$x = A^{-1} \cdot C$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & -2 & 1 \\ 4 & 5 & -2 \\ -2 & -2 & 1 \end{bmatrix}}_{A^{-1}} \underbrace{\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}}_{C}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

$x_1 = 2, \quad x_2 = 0, \quad x_3 = -1$

2. (25 points) Find the corresponding augmented matrix for the following system of linear equations. Use Gauss Jordan elimination to bring the augmented matrix into its reduced row echelon form. Write the solution set for the system. Check your solution set by finding two particular solutions for the system. Determine if the system is consistent, inconsistent, dependent or independent.

$$\begin{cases} x_1 + x_2 + 2x_3 + x_4 + x_5 &= 8 \\ 2x_3 + 2x_4 + 2x_5 &= 8 \\ 2x_1 + 2x_2 + 4x_3 + 2x_4 + 2x_5 &= 16 \\ x_1 + 2x_2 + 3x_3 + x_4 + x_5 &= 12 \end{cases}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 1 & 8 \\ 0 & 0 & 2 & 2 & 2 & 8 \\ 2 & 2 & 4 & 2 & 2 & 16 \\ 1 & 2 & 3 & 1 & 1 & 12 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 1 & 8 \\ 0 & 0 & 2 & 2 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 4 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 1 & 8 \\ 0 & 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 & 8 \end{array}\right]$$

$-2R_1 + R_3 \to R_3$

$-R_1 + R_4 \to R_4$

$R_2 \longleftrightarrow R_4$

$R_4 \leftrightarrow R_3$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 1 & 8 \\ 0 & 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 2 & 2 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$-R_2 + R_1 \to R_1$

$\left(\frac{1}{2}\right) R_3 \to R_3$

$-R_3 + R_1 \to R_1$

$-R_3 + R_2 \to R_2$

Thus;

$x_4 = t \quad , \quad t \in R$

$x_5 = s \quad , \quad s \in R$

$x_1 = 0$

$x_2 = s + t$

$x_3 = 4 - s - t$

System is consistent and dependent.

$SS = \{(0, s+t, 4-s-t, t, s) \mid s, t \in R\}$

Particular Solutions:

$s = 0, \ t = 0 \Rightarrow (0, 0, 4, 0, 0)$

$0 + 0 + 8 + 0 + 0 = 8$ ✓

$2 \cdot 4 + 0 + 0 = 8$ ✓

$0 + 0 + 4 \cdot 4 + 0 + 0 = 16$ ✓

$0 + 0 + 3 \cdot 4 + 0 + 0 = 12$ ✓

$s = 1, \ t = 0 \Rightarrow (0, 1, 3, 0, 1)$

$0 + 1 + 2 \cdot 3 + 0 + 1 = 8$ ✓

$2 \cdot 3 + 2 \cdot 0 + 2 \cdot 1 = 8$ ✓

$2 \cdot 0 + 2 \cdot 1 + 4 \cdot 3 + 2 \cdot 0 + 2 \cdot 1 = 16$ ✓

$0 + 2 \cdot 1 + 3 \cdot 3 + 0 \cdot 1 = 12$ ✓

3. A couple purchased a 180,000 YTL home 5 years ago by paying 20% down and signing a 20 year mortgage at 10.8% compounded monthly.

(a) (6 points) How much did they pay each month ?

$180000 \cdot \frac{20}{100} = 36000$ YTL $\Rightarrow$ $180000 - 36000 = 144000$ YTL

$PV = PMT \, \frac{1-(1+i)^{-n}}{i}$; $n = 20 \times 12 = 240$

$i = \frac{r}{m} = \frac{0.108}{12} = 0.009$

$PV = 144000$ YTL

$144000 = PMT \, \frac{1-(1+0.009)^{-240}}{0.009}$

$PMT = 1466.80$ YTL

(b) (8 points) What is the unpaid balance just after they made the 60^{th} deposit?

60^{th} deposit is after 5 years. Thus unpaid balance is;

$PV = 1466.80 \, \frac{1-(1+0.009)^{-15 \times 12}}{0.009}$

$PV = 130490.11$ YTL

(c) (6 points) Suppose the couple inherit now 50,000 YTL and used it to reduce the unpaid balance just after they made the 60^{th} deposit. What is the new monthly payment if they decide to sign a new 15 year mortgage with the same interest rate?

$130490.11 - 50000 = 80490.11$ $\Rightarrow$ new unpaid balance

$PV = PMT \, \frac{1-(1+i)^{n}}{i}$

$80490.11 = PMT \, \frac{1-(1+0.009)^{-15 \times 12}}{0.009}$

$PMT = 904.77$ YTL

4. You decide to open an account with an investment bank at 13.8% compounded monthly.

(a) (10 points) How much do you have to invest every month for 11 years so that you can withdraw 3000 YTL every month for 9 years? (You put money every month into your account for 11 years, and then withdraw 3000 YTL every month for 9 years)

Withdrawing 3000YTL every month for 9 years:

PMT = 3000, $r = 13.8\% = 0.138$, $m = 12$, $i = \frac{r}{m} = \frac{0.138}{12} = 0.0115$, $t = 9$

$n = t \times m = 9 \times 12 = 108$

$$PV = PMT \frac{1-(1+i)^{-n}}{i} \Rightarrow PV = 3000 \frac{1-(1+0{,}0115)^{-108}}{0.0115} = 184993.01 \text{ YTL}$$

Putting money every month into account for 11 years:

FV = 184993,01, $i = 0.0115$, $t = 11$ years, $n = 11 \cdot 12 = 132$

$$184993.01 = PMT \frac{(1+0.0115)^{132}-1}{0.0115} \Longrightarrow PMT = 603{,}74 \text{ YTL}$$

(b) (7 points) If the investment bank goes bankrupt (iflas) after 17 years (17 years after you opened your account), how much money do you loose (kaybetmek)? (How much money do you still have in your account after 17 years?)

After 11th year, we started to withdraw money from the bank. However we could withdraw just for 6 years. (17 - 11 = 6 years)

Thus, unpaid balance of last 3 year is:

$$PV = 3000 \frac{1-(1+0.0115)^{-12\times 3}}{0.0115} \Rightarrow PV = 88026{,}69 \text{ YTL}$$

(c) (* BONUS* = 5 points) How much interest did you receive in these 17 years?

Interest until 11th year: $184933{,}01 - [(11 \cdot 12) \times 603{,}74] = 105299{,}33$

Interest after 11th year until 17th year:

$= [(6 \cdot 12) \cdot 3000] - [184993.01 - 88026.69]$

$= 216000 - 96966.32 = 119033.68$

Total Interest: $105299{,}33 + 119033.68 = 224333.01$ YTL

5. (a) (10 points) Ali puts 10,000 YTL into an account at 12% compounded monthly. Veli invests 8000 YTL at 15% compounded monthly. After at least how many months does Veli have more money? How much more does he have then?

for Ali: , $P = 10000$ YTL , $r = 12\% = 0.12$, $m = 12$, $i = \frac{0.12}{12} = 0.01$

for Veli: , $P = 8000$ YTL , $r = 15\% = 0.15$, $m = 12$, $i = \frac{0.15}{12} = 0.0125$

$8000(1+0.0125)^n > 10000(1+0.01)^n$

$\left(\frac{1.0125}{1.01}\right)^n > \frac{10}{8} \Rightarrow \ln\left(\frac{1.0125}{1.01}\right)^n > \ln\left(\frac{10}{8}\right)$

$n > 90.26 \Rightarrow n = 91$ months

$8000(1+0.0125)^{91} - 10000(1+0.01)^{91} = 45.19$ YTL

(b) (8 points) An investment increases from 800 YTL to 1100 YTL in 40 days. What is the annual interest rate assuming that the interest is compounded daily? What is the APY?

$A = P(1+i)^n$; $P = 800$ YTL , $A = 1100$ YTL , $n = 40$, $m = 360$, $i = \frac{r}{m} \Rightarrow i = \frac{r}{360}$

$1100 = 800\left(1+\frac{r}{360}\right)^{40} \Rightarrow \frac{11}{8} = \left(1+\frac{r}{360}\right)^{40}$

$\sqrt[40]{11/8} = 1 + \frac{r}{360} \Rightarrow r = 360\left[\sqrt[40]{11/8} - 1\right]$

$r = 2.878 = 287.8\%$

$APY = \left(1+\frac{r}{m}\right)^m - 1$

$APY = \left(1+\frac{2.878}{360}\right)^{360} - 1 \Rightarrow APY = 16.576 = 1657.6\%$

A list of formulas:

$I = Prt$; $A = P(1+rt)$

$A = P(1+i)^n$; $APY = (1+\frac{r}{m})^m - 1$;

$A = Pe^{rt}$; $APY = e^r - 1$;

$FV = PMT\frac{[(1+i)^n - 1]}{i}$

$PV = PMT\frac{[1-(1+i)^{-n}]}{i}$, where $i = \frac{r}{m}$ and $n = mt$