1. (15 points) Solve the following systems of linear equations by **using the inverse matrix method**.

$$a-)\begin{cases} x_1 + x_2 &= 2 \\ 2x_1 + 3x_2 - x_3 &= -1 \\ x_1 + 2x_3 &= 5 \end{cases}$$

$$M = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 2 & 3 & -1 & -1 \\ 1 & 0 & 2 & 5 \end{array}\right]$$

$$MX = C$$

$$\underbrace{M^{-1}.M}_{I}X = M^{-1}.C$$

$$\boxed{X = M^{-1}.C}$$

$$M^{-1}.\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = M^{-1}.\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$$

$$[M|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_3 - R_1 \to R_3 \\ R_2 - 2R_1 \to R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 + R_3 \to R_3 \\ R_1 - R_2 \to R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_2 + R_3 = R_2 \\ R_1 + R_2 = R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -5 & 2 & 1 \\ -3 & 1 & 1 \end{bmatrix} \quad R_1 - R_2 \to R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 6 & -2 & -1 \\ -5 & 2 & 1 \\ -3 & 1 & 1 \end{bmatrix} = [I|M^{-1}]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 6 & -2 & -1 \\ -5 & 2 & 1 \\ -3 & 1 & 1 \end{bmatrix}}_{3\times3} . \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -7 \\ -2 \end{bmatrix}$$

2. Use Gauss Jordan elimination to bring the following augmented matrix into their row reduced echelon form. Write the solution set for the corresponding systems. Find two particular solutions for the system if there exist more than one solution. Determine if the system is **consistent, inconsistent, dependent or independent**.

(a) (15 points) $\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 + x_2 + 2x_3 = 0$
$x_4 = 3$
$x_5 = 4$

let $x_2 = a$
$x_3 = b$

$\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $R_1 - R_2 \to R_1$, $\frac{1}{3}R_3 \to R_3$

$\sim \left[\begin{array}{ccccc|c} 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \to$

$x_1 = -a - 2b$
$x_2 = a$
$x_3 = b$
$x_4 = 3$
$x_5 = 4$ $a, b \in \mathbb{R}$

$\boxed{SS = \{(-a-2b, a, b, 3, 4) \mid a, b \in \mathbb{R}\}}$

Let $(a,b) = (0,0) \to (0,0,0,3,4)$
$(a,b) = (1,0) \to (-1,1,0,3,4)$

The matrix is now in reduced form.

infinitely many solutions
Consistent dependent

(b) (5 points) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$ $\to$ The matrix is in reduced form.
Since the last row produces a contradiction, the system has no solution (inconsistent).

b-) (10 points) Let X be a 3×2 matrix. Find X if

$$X\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

$$X\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -2 & -1 \\ 3 & 0 \end{bmatrix} + 2.\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$X.\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}}_{M} = \begin{bmatrix} 2 & 3 \\ -2 & 1 \\ 5 & 2 \end{bmatrix}$$

$$X.\underbrace{M.M^{-1}}_{I} = \begin{bmatrix} 2 & 3 \\ -2 & 1 \\ 5 & 2 \end{bmatrix}.M^{-1}$$

$$X = \begin{bmatrix} 2 & 3 \\ -2 & 1 \\ 5 & 2 \end{bmatrix} M^{-1}$$

$$M.M^{-1} = I$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left.\begin{aligned} 2a+b&=1 \\ 2c+d&=0 \\ a+b&=0 \\ c+d&=1 \end{aligned}\right\} \quad \begin{aligned} a&=1 \\ b&=-1 \\ c&=-1 \\ d&=2 \end{aligned}$$

$$X = \begin{bmatrix} 2 & 3 \\ -2 & 1 \\ 5 & 2 \end{bmatrix}_{3\times2} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}_{2\times2}$$

$$X = \begin{bmatrix} -1 & 4 \\ -3 & 4 \\ 3 & -1 \end{bmatrix} //$$

A list of formulas: $I = Prt$; $A = P(1 + rt)$

$A = P(1+i)^n$; $APY = (1 + \frac{r}{m})^m - 1$; $A = Pe^{rt}$; $APY_{cont} = e^r - 1$;

$FV = PMT\frac{[(1+i)^n - 1]}{i}$; $PV = PMT\frac{[1-(1+i)^{-n}]}{i}$, where $i = \frac{r}{m}$ and $n = mt$

3. (a) (10 points) Which of the following options would you choose to invest your money?

- (option 1) 7.8% compounded annually
- (option 2) 7.4% compounded continuously
- (option 3) 7.5% compounded quarterly

A = future value

P = present value

- 1st option: $APY_1 = 0{,}078 \rightarrow$ $\boxed{\text{best option}}$
- 2nd option: $APY_2 = e^{0{,}074} - 1 = 0{,}0768$
- 3rd option: $APY_3 = \left(1 + \frac{0{,}075}{4}\right)^4 - 1 = 0{,}0771$

(b) (10 points) If you choose (option 3) how many years will it take for your money to triple?

A = 3P → future value

P = present value

n = needed years to triple present value

$i = \frac{0{,}75}{4} = 0{,}01875$

$3P = P.(1 + 0{,}01875)^n$

$3 = (1{,}01875)^n$

$\ln 3 = n.\ln(1{,}01875)$

$n = \frac{\ln 3}{\ln(1{,}01875)} \cong 59{,}19$

we need 60 quaters

$\boxed{15 \text{ years}}$

4. (15 points) Serpil put 100TL every month in a retirement fund for a year. Then she got promoted and increased the money she put into the account to 150TL for the next two years. How much will she have in the account at the end of this 3 year period if the account earns 9% compounded monthly.

$$i = \frac{r}{m} = \frac{0{,}09}{12} = \frac{3}{400} = 0{,}0075$$

For first year: $FV = PMT \dfrac{(1+i)^n - 1}{i}$

$$= 100 \cdot \frac{(1 + 0{,}0075)^{12} - 1}{0{,}0075}$$

$$= 1{,}250.76$$

$$A = 1250.76\,(1 + 0{,}0075)^{24} = 1496.43$$

For 2^{nd} and 3^{rd} year $= 150 \cdot \dfrac{(1+0{,}0075)^{24} - 1}{0{,}0075}$

$$= 3{,}928.27 \text{ TL}$$

Total $= 3928{,}27 + 1496.43 = 5{,}424{,}70$ TL

5. A couple purchased a house today for 100.000TL signing a 20-year mortgage at 12% compounded monthly.

(a) (10 points) If they sell the house ten years later for 120.000TL, how much would they receive after paying off their debt to the bank?

$P.V = 100.000TL$ $\quad n = 12.20 \quad \boxed{n = 240} \quad r = 12\% \quad \boxed{r = 0{,}12} \quad i = \frac{0{,}12}{12} \quad \boxed{i = 0{,}01}$

$PMT = 100.000 \cdot \frac{0{,}01}{1-(1+0{,}01)^{-240}} = 1{,}101.09$ (per month)

unpaid Balance after 10 years $\rightarrow 1{,}101.09 \times \frac{1-(1+0{,}01)^{-120}}{0{,}01} = 76{,}746.55$

received money after paying off their dept to the bank $\rightarrow 120.000 - 76{,}746{,}55 = 43{,}253.45$

(b) (10 points) How much interest did they pay in the tenth year?

Unpaid balance 9th year $= 1101.09 \dfrac{1-\left(1+\frac{0{,}12}{12}\right)^{-132}}{\frac{0{,}12}{12}}$

$= 80.501.52$

Unpaid balance reduction in the tenth year

$80501.52 - 76746.55 = 3754.97$

Total payment $= 12 \times 1101.09 = 13{,}213.08$

Interest paid in tenth year $= 13\,213{,}08 - 3.754.97 =$

$\boxed{9458.11}$