KOÇ UNIVERSITY

MATH 101 - FINITE MATHEMATICS

Final January 12, 2016

Duration of Exam: 120 minutes

INSTRUCTIONS: You can NOT use calculators in the exam. No books, no notes, and no talking allowed. You must always **explain your answers and show your work** to receive **full credit**. Use the back of these pages if necessary. **Print (use CAPITAL LETTERS) and sign your name, and indicate your section below.**

Name: ________________

Surname: SOLUTION

Signature: ________________

Section (Check One):

Lecture 1: Mine Çağlar M-W (10:00) ___
Lecture 2: Mine Çağlar M-W (13:00) ___
Lecture 3: Ayberk Zeytin Tu-Th(13:00) ___
Lecture 4: Ayberk Zeytin Tu-Th(16:00) ___

PROBLEM	POINTS	SCORE
1	15	
2	15	
3	17	
4	32	
5	13	
6	13	
TOTAL	105	

1. (15 points) Aylin, Berna and Ceren each borrow the same loan P for 5 years at the same interest rate r, compounded semi-annually. Let D_A, D_B, and D_C denote the total payment made by Aylin, Berna and Ceren, respectively.

(a) (4 points) Aylin repays her total debt (principal and interest) in one payment at the end of 5 years. Write down an expression for D_A, using the appropriate finance formula.

$$D_A = P\left(1+\frac{r}{2}\right)^{(5)(2)} = P\left(1+\frac{r}{2}\right)^{10}$$

(b) (4 points) Berna repays only the interest at the end of each 6-month period as it accrues (= builds up). She repays the principal at the end of 5 years. Write down an expression for D_B, using the appropriate finance formula.

$I = P.\frac{r}{2}$ is the interest at the end of the first 6 months and the same interest at the end of each 6 months because interest does not accrue interest, it is paid!

$$D_B = 10.P.(r/2) + P = P + 5Pr$$

(c) (4 points) Ceren repays her total debt in 10 equal payments made at the end of each 6-month period. Write down an expression for D_C, using the appropriate finance formula.

$$P = PMT\,\frac{1-(1+r/2)^{-10}}{r/2} \Rightarrow \text{find } PMT.$$

$$\Rightarrow \quad D_C = 10(PMT) = 10.P.\frac{r/2}{1-(1+r/2)^{-10}}$$

(d) (3 points) Order D_A, D_B and D_C increasingly by choosing one of i)-vi) below.

i) $D_A < D_B < D_C$
ii) $D_A < D_C < D_B$
iii) $D_B < D_C < D_A$
iv) $D_B < D_A < D_C$
v) $D_C < D_B < D_A$
vi) $D_C < D_A < D_B$

Explain your choice.

In (a), all is paid in the end so the "highest" interest must build up.
In (b), the principal continues to be a debt all the time, and interest is calculated over it.
In (c), some of the principal is repaid within each equal PMT (recall amortization schedule), so it must build up the "least" interest over the 5 years.

A list of formulas: $I = Prt$; $A = P(1+rt)$

$A = P(1+i)^n$; $APY = (1+\frac{r}{m})^m - 1$; $A = Pe^{rt}$; $APY = e^r - 1$;

$FV = PMT\frac{[(1+i)^n-1]}{i}$; $PV = PMT\frac{[1-(1+i)^{-n}]}{i}$, where $i = \frac{r}{m}$ and $n = mt$

2. (a) (10 points) Compute the determinant of the matrix

$$M = \begin{pmatrix} a & 0 & 0 & b \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ c & 0 & 0 & d \end{pmatrix}$$

$$\det M = (-1)^{1+1} a \begin{vmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & d \end{vmatrix} + (-1)^{1+4} b \begin{vmatrix} 0 & a & b \\ 0 & c & d \\ c & 0 & 0 \end{vmatrix}$$

$$= a\,d\,(-1)^{1+3} \begin{vmatrix} a & b \\ c & d \end{vmatrix} - b\,c\,(-1)^{3+1} \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$= ad\,(ad-bc) - bc\,(ad-bc)$$

$$= (ad-bc)^2$$

(b) (5 points) Find all $\theta \in [101\pi, 102\pi]$ for which the following matrix is not invertible

$$M = \begin{pmatrix} \cot(\theta) & 0 & 0 & \tan(\theta) \\ 0 & \cot(\theta)^{\,a} & \tan(\theta)^{\,b} & 0 \\ 0 & \cos(\theta)_{c} & \sin(\theta)_{d} & 0 \\ \cos(\theta) & 0 & 0 & \sin(\theta) \end{pmatrix}.$$

(Hint: Use (a).)

$$(ad-bc)^2 = (\cot\theta\sin\theta - \tan\theta\cos\theta)^2 = 0$$

$$\Rightarrow \quad \cot\theta\sin\theta - \tan\theta\cos\theta = 0$$

$$\frac{\cos\theta}{\cancel{\sin\theta}}\cancel{\sin\theta} - \frac{\sin\theta}{\cancel{\cos\theta}}\cancel{\cos\theta} = 0.$$

$$\cos\theta - \sin\theta = 0.$$

$$\cos\theta = \sin\theta$$

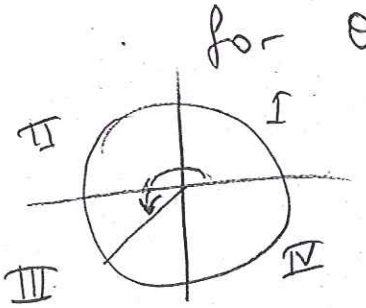
$$\Rightarrow \quad \theta = \begin{cases} \pi/4 + 2\pi k & \text{Quadrant I} \\ 5\pi/4 + 2\pi k & \text{Quad. III} \end{cases}$$

for $\theta \in [101\pi, 102\pi]$

$\downarrow \qquad \downarrow$

$[50(2\pi)+\pi,\ 51(2\pi)]$

in quadrant III



and $101\pi + \pi/4 = 101.25\pi$

3. (a) (12 points) Solve the following two systems:

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 5 \\ x_1 - x_2 &= -1 \\ -x_1 + x_2 + x_3 &= 5 \end{aligned} \qquad \text{and} \qquad \begin{aligned} y_1 - 2y_2 + 2y_3 &= 1 \\ y_1 - y_2 &= -1 \\ -y_1 + y_2 + y_3 &= 4 \end{aligned}$$

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \quad AX = \begin{bmatrix} 5 \\ -1 \\ 5 \end{bmatrix} \qquad AY = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1+R_3\to R_3]{R_1+R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & -1 & 3 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_2+R_3\to R_3]{2R_2+R_1\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 2 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \xrightarrow[2R_3+R_2\to R_2]{2R_3+R_1\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 4 & 2 \\ 0 & 1 & 0 & -1 & 3 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} -1 & 4 & 2 \\ -1 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}}_{A^{-1}}$$

$$X = \begin{bmatrix} -1 & 4 & 2 \\ -1 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \qquad Y = \begin{bmatrix} -1 & 4 & 2 \\ -1 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$$

(b) (5 points) Use $\begin{pmatrix} 5 \\ -1 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$ to solve the system:

$$\underbrace{\begin{aligned} z_1 - 2z_2 + 2z_3 &= 4 \\ z_1 - z_2 &= 0 \\ -z_1 + z_2 + z_3 &= 1 \end{aligned}}$$

$$AZ = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow AZ = \begin{bmatrix} 5 \\ -1 \\ 5 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

$$Z = A^{-1}\begin{bmatrix} 5 \\ -1 \\ 5 \end{bmatrix} - A^{-1}\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}}^{X} - \underbrace{\begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}}^{Y}$$

$$= \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$$

4. Consider the following 2-player zero sum matrix game:

$$\begin{pmatrix} 1 & -1 & -3 \\ -1 & 1 & 2 \\ 2 & -1 & -2 \end{pmatrix}$$

(a) (5 points) Reduce the game to a 2×3 game.

$1 \leq 2 \quad -1 \leq -1 \quad -3 \leq -2$, first row is recessive.

$$\Rightarrow M = \begin{bmatrix} -1 & 1 & 2 \\ 2 & -1 & -2 \end{bmatrix}$$

(b) (7 points) Using the matrix you obtained in part (a) write the corresponding maximization and minimization (i.e. linear programming) problems.

Need positive matrix $\Rightarrow$ add 3

$$M_1 = \begin{bmatrix} 2 & 4 & 5 \\ 5 & 2 & 1 \end{bmatrix}$$

Minimize $y = x_1 + x_2$

s.t.

$$2x_1 + 5x_2 \geq 1$$
$$4x_1 + 2x_2 \geq 1$$
$$5x_1 + x_2 \geq 1$$
$$x_1, x_2 \geq 0$$

Maximize $y = z_1 + z_2 + z_3$

s.t.

$$2z_1 + 4z_2 + 5z_3 \leq 1$$
$$5z_1 + 2z_2 + z_3 \leq 1$$
$$z_1, z_2, z_3 \geq 0$$

(c) (20 points) Solve the game.

$$\begin{array}{c} \\ x_1 \\ x_2 \\ y \end{array}\begin{array}{c} \begin{array}{cccccc|c} z_1 & z_2 & z_3 & x_1 & x_2 & y & \end{array} \\ \left[\begin{array}{cccccc|c} 2 & 4 & 5 & 1 & 0 & 0 & 1 \\ 5 & 2 & 1 & 0 & 1 & 0 & 1 \\ \hline -1 & -1 & -1 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \begin{array}{c} 1/2 \\ 1/5 \\ \\ \end{array}$$

$$\xrightarrow{\frac{1}{5}R_2 \to R_2} \sim \left[\begin{array}{cccccc|c} 2 & 4 & 5 & 1 & 0 & 0 & 1 \\ 1 & 2/5 & 1/5 & 0 & 1/5 & 0 & 1/5 \\ \hline -1 & -1 & -1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_2 + R_3 \to R_3]{-2R_2 + R_1 \to R_1} \sim \begin{array}{c} x_1 \\ z_1 \\ y \end{array}\left[\begin{array}{cccccc|c} 0 & 16/5 & 23/5 & 1 & -2/5 & 0 & 3/5 \\ 1 & 2/5 & 1/5 & 0 & 1/5 & 0 & 1/5 \\ \hline 0 & -3/5 & -4/5 & 0 & 1/5 & 1 & 1/5 \end{array}\right] \begin{array}{c} 3/23 \\ 1 \\ \\ \end{array}$$

$$\xrightarrow{\frac{5}{23}R_1 \to R_1} \sim \left[\begin{array}{cccccc|c} 0 & 16/23 & 1 & 5/23 & -2/23 & 0 & 3/23 \\ 1 & 2/5 & 1/5 & 0 & 1/5 & 0 & 1/5 \\ \hline 0 & -3/5 & -4/5 & 0 & 1/5 & 1 & 1/5 \end{array}\right]$$

$$\xrightarrow[\frac{4}{5}R_1 + R_3 \to R_3]{-\frac{1}{5}R_1 + R_2 \to R_2} \begin{array}{c} z_3 \\ z_1 \\ y \end{array}\left[\begin{array}{cccccc|c} 0 & 16/23 & 1 & 5/23 & -2/23 & 0 & 3/23 \\ 1 & 6/23 & 0 & -1/23 & 5/23 & 0 & 4/23 \\ \hline 0 & -1/23 & 0 & 4/23 & 3/23 & 1 & 7/23 \end{array}\right]$$

$$\xrightarrow{\frac{23}{16}R_1 \to R_1} \sim \left[\begin{array}{cccccc|c} 0 & 1 & 23/16 & 5/16 & -1/8 & 0 & 3/16 \\ 1 & 6/23 & 0 & -1/23 & 5/23 & 0 & 4/23 \\ \hline 0 & -1/23 & 0 & 4/23 & 3/23 & 1 & 7/23 \end{array}\right] \xrightarrow[\frac{1}{23}R_1 + R_3 \to R_3]{-\frac{6}{23}R_1 + R_2 \to R_2}$$

$$\begin{array}{c} \\ z_2 \\ z_1 \\ y \end{array}\begin{array}{c} \begin{array}{cccccc|c} z_1 & z_2 & z_3 & x_1 & x_2 & y & \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 1 & 23/16 & 5/16 & -1/8 & 0 & 3/16 \\ 1 & 0 & -3/8 & -1/8 & 1/4 & 0 & 1/8 \\ \hline 0 & 0 & 1/16 & 3/16 & 1/8 & 1 & 5/16 \end{array}\right] \end{array}$$

From the final tableau:

$z_1 = 7/40$

$z_2 = 3/16$

$z_3 = 0$

$v_1 = \frac{16}{5} \left(= \frac{1}{y}\right)$

$\Rightarrow \boxed{v = \frac{16}{5} - 3 = \frac{1}{5}}$

$q_1 = \frac{1}{8} \cdot 16/5 = \frac{2}{5}$

$q_2 = \frac{3}{16} \cdot \frac{16}{5} = \frac{3}{5}$

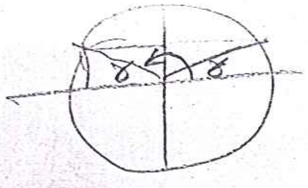
$q_3 = 0$

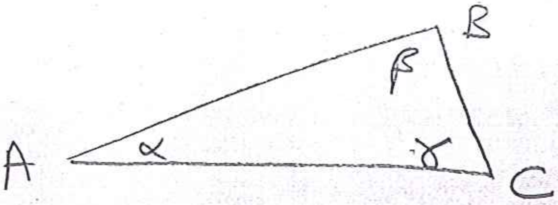
Similarly,

$p_1 = \frac{3}{5}$, $p_2 = \frac{2}{5}$

since $x_1 = 3/16$

$x_2 = 1/8$





$\alpha+\beta=\pi-\gamma$

$\sin(\alpha+\beta)=\sin(\pi-\gamma)$

$=\sin\gamma.$

5. (a) (7 points) Show that if α, β and γ are the three angles of a triangle, then we have:

$$\tan(\alpha)+\tan(\beta)+\tan(\gamma)=\tan(\alpha)\tan(\beta)\tan(\gamma).$$

$$\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}+\frac{\sin\gamma}{\cos\gamma}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}+\frac{\sin\gamma}{\cos\gamma}$$

$$=\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}+\frac{\sin\gamma}{\cos\gamma}$$

$$=\frac{\sin\gamma}{\cos\alpha\cos\beta}+\frac{\sin\gamma}{\cos\gamma}=\frac{\sin\gamma(\cos\gamma+\cos\alpha\cos\beta)}{\cos\alpha\cos\beta\cos\gamma}$$

$\cos(\alpha+\beta)=\cos\alpha.\cos\beta-\sin\alpha\sin\beta$

$$=\tan\gamma.\frac{(-\cos(\alpha+\beta)+\cos\alpha\cos\beta)}{\cos\alpha.\cos\beta}$$

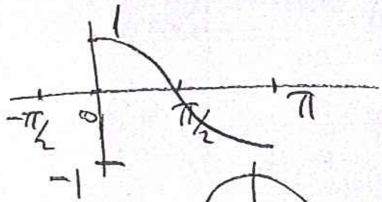
$$=\tan\gamma\ \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}=\tan\gamma.\tan\alpha.\tan\beta$$

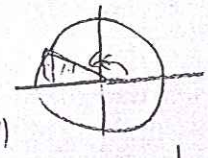
(b) (6 points) Find

i) $\operatorname{arcsec}(-2/\sqrt{3}))$ ii) $\arctan(-1\cancel{/\sqrt{2}})$ iii) $\arcsin(-1/2)$

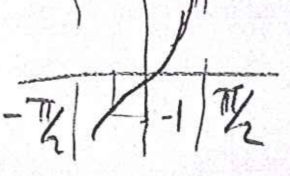
i) $\sec\theta=\frac{-2}{\sqrt{3}} \Rightarrow \cos\theta=\frac{-\sqrt{3}}{2}$

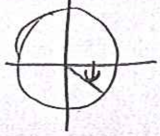
$\Rightarrow \theta=\pi-\frac{\pi}{6}=5\pi/6$ //



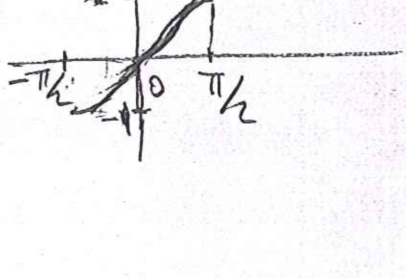


ii) $\tan\theta=-1 \Rightarrow \theta=-\pi/4$ //





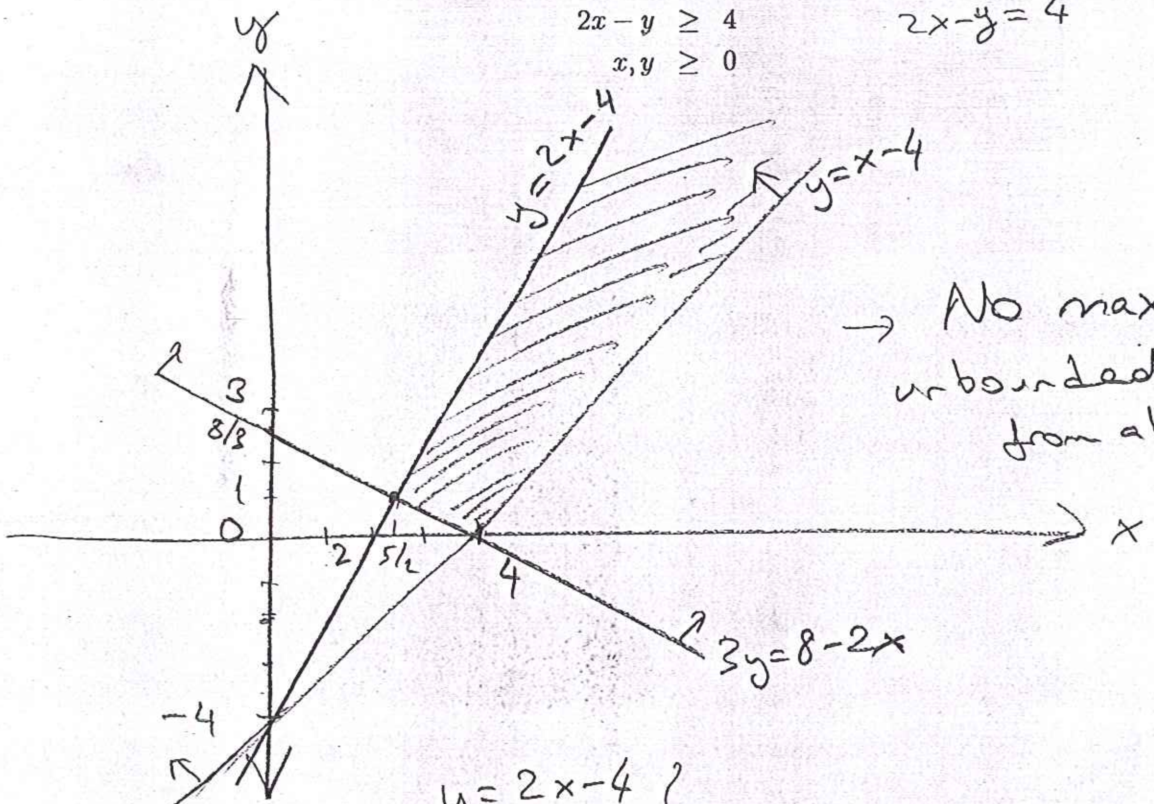
iii) $\sin\theta=-\frac{1}{2} \Rightarrow \theta=-\pi/6$ //



6. (13 points) Minimize and maximize $P = 4x + 6y$ subject to

$$\begin{aligned} x - y &\le 4 \\ 2x + 3y &\ge 8 \\ 2x - y &\ge 4 \\ x, y &\ge 0 \end{aligned}$$

$x - y = 4$
$2x + 3y = 8$
$2x - y = 4$



→ No maximum
unbounded from above.

$$\begin{aligned} y &= 2x - 4 \\ + \quad 3y &= 8 - 2x \\ \hline 4y &= 4 \end{aligned} \Rightarrow y = 1 \Rightarrow x = 5/2$$

Corner Points

x	y	P
5/2	1	10 + 6 = 16
4	0	16

Both $x = 5/2, y = 1$
and $x = 4, y = 0$
and all points joining
$(5/2, 1)$ and $(4, 0)$ are
optimal solutions.

Minimum $P = 16$