Homework 11 – Solutions

Section 5.4

7. $f(t) = \frac{1}{t^3+1}$ and $g(x) = \int_1^x \frac{1}{t^3+1}\,dt$, so by FTC1, $g'(x) = f(x) = \frac{1}{x^3+1}$. Note that the lower limit, 1, could be any real number greater than -1 and not affect this answer.

8. $f(t) = e^{t^2-t}$ and $g(x) = \int_3^x e^{t^2-t}\,dt$, so by FTC1, $g'(x) = f(x) = e^{x^2-x}$.

13. Let $u = \frac{1}{x}$. Then $\frac{du}{dx} = -\frac{1}{x^2}$. Also, $\frac{dh}{dx} = \frac{dh}{du}\frac{du}{dx}$, so

$$h'(x) = \frac{d}{dx}\int_2^{1/x} \arctan t\,dt = \frac{d}{du}\int_2^u \arctan t\,dt \cdot \frac{du}{dx} = \arctan u\,\frac{du}{dx} = -\frac{\arctan(1/x)}{x^2}.$$

14. Let $u = x^2$. Then $\frac{du}{dx} = 2x$. Also, $\frac{dh}{dx} = \frac{dh}{du}\frac{du}{dx}$, so

$$h'(x) = \frac{d}{dx}\int_0^{x^2} \sqrt{1+r^3}\,dr = \frac{d}{du}\int_0^u \sqrt{1+r^3}\,dr \cdot \frac{du}{dx} = \sqrt{1+u^3}(2x) = 2x\sqrt{1+(x^2)^3} = 2x\sqrt{1+x^6}.$$

25. By FTC2, $\int_1^4 f'(x)\,dx = f(4) - f(1)$, so $17 = f(4) - 12 \Rightarrow f(4) = 17 + 12 = 29$.

Section 5.5

2. Let $u = 2 + x^4$. Then $du = 4x^3\,dx$ and $x^3\,dx = \frac{1}{4}\,du$,

so $\int x^3(2+x^4)^5\,dx = \int u^5\left(\frac{1}{4}\,du\right) = \frac{1}{4}\frac{u^6}{6} + C = \frac{1}{24}(2+x^4)^6 + C$.

4. Let $u = 1 - 6t$. Then $du = -6\,dt$ and $dt = -\frac{1}{6}\,du$, so

$$\int \frac{dt}{(1-6t)^4} = \int \frac{-\frac{1}{6}\,du}{u^4} = -\frac{1}{6}\int u^{-4}\,du = -\frac{1}{6}\frac{u^{-3}}{-3} + C = \frac{1}{18u^3} + C = \frac{1}{18(1-6t)^3} + C.$$

7. Let $u = x^2$. Then $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$, so $\int x\sin(x^2)\,dx = \int \sin u\left(\frac{1}{2}\,du\right) = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos(x^2) + C$.

8. Let $u = x^3 + 5$. Then $du = 3x^2\,dx$ and $x^2\,dx = \frac{1}{3}\,du$, so

$\int x^2(x^3+5)^9\,dx = \int u^9\left(\frac{1}{3}\,du\right) = \frac{1}{3}\cdot\frac{1}{10}u^{10} + C = \frac{1}{30}(x^3+5)^{10} + C$.

12. Let $u = e^x$. Then $du = e^x\,dx$, so $\int e^x\cos(e^x)\,dx = \int \cos u\,du = \sin u + C = \sin(e^x) + C$.

13. Let $u = \ln x$. Then $du = \frac{dx}{x}$, so $\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(\ln x)^3 + C$.

14. Let $u = x^2 + 1$. Then $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$, so

$$\int \frac{x}{(x^2+1)^2}\,dx = \int u^{-2}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\cdot\frac{-1}{u} + C = \frac{-1}{2u} + C = \frac{-1}{2(x^2+1)} + C.$$

23. Let $u = x^3 + 3x$. Then $du = (3x^2 + 3)\,dx$ and $\frac{1}{3}\,du = (x^2 + 1)\,dx$, so

$\int (x^2+1)(x^3+3x)^4\,dx = \int u^4 \left(\frac{1}{3}\,du\right) = \frac{1}{3} \cdot \frac{1}{5}u^5 + C = \frac{1}{15}(x^3+3x)^5 + C.$

32. Let $u = e^x + 1$. Then $du = e^x\,dx$, so $\displaystyle\int \frac{e^x}{e^x+1}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln(e^x+1) + C.$

34. Let $u = \cos x$. Then $du = -\sin x\,dx$ and $\sin x\,dx = -du$, so

$$\int \frac{\sin x}{1+\cos^2 x}\,dx = \int \frac{-du}{1+u^2} = -\tan^{-1} u + C = -\tan^{-1}(\cos x) + C.$$

43. Let $u = 1 + 7x$, so $du = 7\,dx$. When $x = 0$, $u = 1$; when $x = 1$, $u = 8$. Thus,

$$\int_0^1 \sqrt[3]{1+7x}\,dx = \int_1^8 u^{1/3}\left(\tfrac{1}{7}\,du\right) = \tfrac{1}{7}\left[\tfrac{3}{4}u^{4/3}\right]_1^8 = \tfrac{3}{28}\left(8^{4/3} - 1^{4/3}\right) = \tfrac{3}{28}(16-1) = \tfrac{45}{28}$$

48. Let $u = \sin x$, so $du = \cos x\,dx$. When $x = 0$, $u = 0$; when $x = \frac{\pi}{2}$, $u = 1$. Thus,

$\int_0^{\pi/2} \cos x\, \sin(\sin x)\,dx = \int_0^1 \sin u\,du = \big[-\cos u\big]_0^1 = -(\cos 1 - 1) = 1 - \cos 1.$

55. Let $u = \ln x$, so $du = \dfrac{dx}{x}$. When $x = e$, $u = 1$; when $x = e^4$; $u = 4$. Thus,

$$\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} = \int_1^4 u^{-1/2}\,du = 2\left[u^{1/2}\right]_1^4 = 2(2-1) = 2.$$

Section 5.6

1. Let $u = \ln x$, $dv = x^2\,dx \;\Rightarrow\; du = \frac{1}{x}\,dx$, $v = \frac{1}{3}x^3$. Then by Equation 2,

$\int x^2 \ln x\,dx = (\ln x)\left(\frac{1}{3}x^3\right) - \int \left(\frac{1}{3}x^3\right)\left(\frac{1}{x}\right)dx = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2\,dx = \frac{1}{3}x^3 \ln x - \frac{1}{3}\left(\frac{1}{3}x^3\right) + C$

$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C \quad \left[\text{or } \frac{1}{3}x^3\left(\ln x - \frac{1}{3}\right) + C\right]$

2. Let $u = \theta$, $dv = \cos\theta\,d\theta \;\Rightarrow\; du = d\theta$, $v = \sin\theta$. Then by Equation 2,

$\int \theta\cos\theta\,d\theta = \theta\sin\theta - \int \sin\theta\,d\theta = \theta\sin\theta + \cos\theta + C.$

5. Let $u = r$, $dv = e^{r/2}\,dr \;\Rightarrow\; du = dr$, $v = 2e^{r/2}$.

Then $\int re^{r/2}\,dr = 2re^{r/2} - \int 2e^{r/2}\,dr = 2re^{r/2} - 4e^{r/2} + C = 2(r-2)e^{r/2} + C.$

6. Let $u = t$, $dv = \sin 2t\,dt \;\Rightarrow\; du = dt$, $v = -\frac{1}{2}\cos 2t$. Then

$\int t\sin 2t\,dt = -\frac{1}{2}t\cos 2t + \frac{1}{2}\int \cos 2t\,dt = -\frac{1}{2}t\cos 2t + \frac{1}{4}\sin 2t + C.$

9. Let $u = \ln\sqrt[3]{x}$, $dv = dx \;\Rightarrow\; du = \dfrac{1}{\sqrt[3]{x}}\left(\dfrac{1}{3}x^{-2/3}\right)dx = \dfrac{1}{3x}\,dx$, $v = x$. Then

$$\int \ln\sqrt[3]{x}\,dx = x\ln\sqrt[3]{x} - \int x \cdot \frac{1}{3x}\,dx = x\ln\sqrt[3]{x} - \frac{1}{3}x + C.$$

Second solution: Rewrite $\int \ln\sqrt[3]{x}\,dx = \frac{1}{3}\int \ln x\,dx$, and apply Example 2.

Third solution: Substitute $y = \sqrt[3]{x}$, to obtain $\int \ln\sqrt[3]{x}\,dx = 3\int y^2 \ln y\,dy$, and apply Exercise 1.

16. First let $u = x^2 + 1$, $dv = e^{-x}\,dx \Rightarrow du = 2x\,dx$, $v = -e^{-x}$. By (6),

$\int_0^1 (x^2+1)e^{-x}\,dx = \left[-(x^2+1)e^{-x}\right]_0^1 + \int_0^1 2xe^{-x}\,dx = -2e^{-1} + 1 + 2\int_0^1 xe^{-x}\,dx.$

Next let $U = x$, $dV = e^{-x}\,dx \Rightarrow dU = dx$, $V = -e^{-x}$. By (6) again,

22. Let $u = r^2$, $dv = \dfrac{r}{\sqrt{4+r^2}}\,dr \Rightarrow du = 2r\,dr$, $v = \sqrt{4+r^2}$. By (6),

$$\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr = \left[r^2\sqrt{4+r^2}\right]_0^1 - 2\int_0^1 r\sqrt{4+r^2}\,dr = \sqrt{5} - \tfrac{2}{3}\left[(4+r^2)^{3/2}\right]_0^1$$

$$= \sqrt{5} - \tfrac{2}{3}(5)^{3/2} + \tfrac{2}{3}(8) = \sqrt{5}\left(1 - \tfrac{10}{3}\right) + \tfrac{16}{3} = \tfrac{16}{3} - \tfrac{7}{3}\sqrt{5}$$

23. Let $u = (\ln x)^2$, $dv = dx \Rightarrow du = \dfrac{2}{x}\ln x\,dx$, $v = x$. By (6), $I = \int_1^2 (\ln x)^2\,dx = \left[x(\ln x)^2\right]_1^2 - 2\int_1^2 \ln x\,dx$.

To evaluate the last integral, let $U = \ln x$, $dV = dx \Rightarrow dU = \dfrac{1}{x}\,dx$, $V = x$. Thus,

$$I = \left[x(\ln x)^2\right]_1^2 - 2\left(\left[x\ln x\right]_1^2 - \int_1^2 dx\right) = \left[x(\ln x)^2 - 2x\ln x + 2x\right]_1^2$$

$$= \left(2(\ln 2)^2 - 4\ln 2 + 4\right) - (0 - 0 + 2) = 2(\ln 2)^2 - 4\ln 2 + 2$$

24. Let $u = \sin(t-s)$, $dv = e^s\,ds \Rightarrow du = -\cos(t-s)\,ds$, $v = e^s$. Then

$I = \int_0^t e^s \sin(t-s)\,ds = \left[e^s\sin(t-s)\right]_0^t + \int_0^t e^s\cos(t-s)\,ds = e^t\sin 0 - e^0\sin t + I_1$. For I_1, let $U = \cos(t-s)$,

$dV = e^s\,ds \Rightarrow dU = \sin(t-s)\,ds$, $V = e^s$. So $I_1 = \left[e^s\cos(t-s)\right]_0^t - \int_0^t e^s\sin(t-s)\,ds = e^t\cos 0 - e^0\cos t - I$.

Thus, $I = -\sin t + e^t - \cos t - I \Rightarrow 2I = e^t - \cos t - \sin t \Rightarrow I = \tfrac{1}{2}(e^t - \cos t - \sin t)$.

25. Let $y = \sqrt{x}$, so that $dy = \tfrac{1}{2}x^{-1/2}\,dx = \dfrac{1}{2\sqrt{x}}\,dx = \dfrac{1}{2y}\,dx$. Thus, $\int \cos\sqrt{x}\,dx = \int \cos y\,(2y\,dy) = 2\int y\cos y\,dy$. Now

use parts with $u = y$, $dv = \cos y\,dy$, $du = dy$, $v = \sin y$ to get $\int y\cos y\,dy = y\sin y - \int \sin y\,dy = y\sin y + \cos y + C_1$,

so $\int \cos\sqrt{x}\,dx = 2y\sin y + 2\cos y + C = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$.

30. Let $y = \ln x$, so that $dy = \frac{1}{x}\,dx \Rightarrow dx = x\,dy = e^y\,dy$. Thus,

$\int \sin(\ln x)\,dx = \int \sin y\, e^y\,dy = \tfrac{1}{2}e^y(\sin y - \cos y) + C$ [by Example 4] $= \tfrac{1}{2}x[\sin(\ln x) - \cos(\ln x)] + C$.

45. For $I = \int_1^4 xf''(x)\,dx$, let $u = x$, $dv = f''(x)\,dx \Rightarrow du = dx$, $v = f'(x)$. Then

$I = \left[xf'(x)\right]_1^4 - \int_1^4 f'(x)\,dx = 4f'(4) - 1\cdot f'(1) - [f(4) - f(1)] = 4\cdot 3 - 1\cdot 5 - (7-2) = 12 - 5 - 5 = 2.$

We used the fact that f'' is continuous to guarantee that I exists.