Homework 12 – Solutions

Section 5.7

1. $\int \sin^3 x \cos^2 x\,dx = \int \sin^2 x \cos^2 x \sin x\,dx = \int (1-\cos^2 x)\cos^2 x \sin x\,dx$

$= \int (1-u^2)u^2(-du) \qquad [u=\cos x,\ du=-\sin x\,dx]$

$= \int (u^2-1)u^2\,du = \int (u^4-u^2)\,du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$

2. $\int_0^{\pi/2} \cos^5 x\,dx = \int_0^{\pi/2} (\cos^2 x)^2 \cos x\,dx = \int_0^{\pi/2} (1-\sin^2 x)^2 \cos x\,dx$

$= \int_0^1 (1-u^2)^2\,du \qquad [u=\sin x,\ du=\cos x\,dx]$

$= \int_0^1 (1-2u^2+u^4)\,du = \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_0^1 = \left(1-\frac{2}{3}+\frac{1}{5}\right) - 0 = \frac{8}{15}$

6. $\int_0^{\pi/2} \sin^2 x \cos^2 x\,dx = \int_0^{\pi/2} \frac{1}{4}(4\sin^2 x \cos^2 x)\,dx = \int_0^{\pi/2} \frac{1}{4}(2\sin x \cos x)^2 dx = \frac{1}{4}\int_0^{\pi/2} \sin^2 2x\,dx$

$= \frac{1}{4}\int_0^{\pi/2} \frac{1}{2}(1-\cos 4x)\,dx = \frac{1}{8}\int_0^{\pi/2} (1-\cos 4x)\,dx = \frac{1}{8}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi/2} = \frac{1}{8}\left(\frac{\pi}{2}\right) = \frac{\pi}{16}$

21. $\dfrac{5x+1}{(2x+1)(x-1)} = \dfrac{A}{2x+1} + \dfrac{B}{x-1}$. Multiply both sides by $(2x+1)(x-1)$ to get $5x+1 = A(x-1) + B(2x+1)$ $\Rightarrow$

$5x+1 = Ax - A + 2Bx + B$ $\Rightarrow$ $5x+1 = (A+2B)x + (-A+B)$.

The coefficients of x must be equal and the constant terms are also equal, so $A + 2B = 5$ and

$-A + B = 1$. Adding these equations gives us $3B = 6$ $\Leftrightarrow$ $B = 2$, and hence, $A = 1$. Thus,

$$\int \frac{5x+1}{(2x+1)(x-1)}\,dx = \int \left(\frac{1}{2x+1} + \frac{2}{x-1}\right)dx = \tfrac{1}{2}\ln|2x+1| + 2\ln|x-1| + C.$$

Another method: Substituting 1 for x in the equation $5x+1 = A(x-1) + B(2x+1)$ gives $6 = 3B$ $\Leftrightarrow$ $B = 2$.

Substituting $-\frac{1}{2}$ for x gives $-\frac{3}{2} = -\frac{3}{2}A$ $\Leftrightarrow$ $A = 1$.

22. $\dfrac{x-4}{x^2-5x+6} = \dfrac{A}{x-2} + \dfrac{B}{x-3}$. Multiply both sides by $(x-2)(x-3)$ to get $x-4 = A(x-3) + B(x-2)$ $\Rightarrow$

$x - 4 = Ax - 3A + Bx - 2B$ $\Rightarrow$ $x - 4 = (A+B)x + (-3A - 2B)$.

The coefficients of x must be equal and the constant terms are also equal, so $A + B = 1$ and $-3A - 2B = -4$.

Adding twice the first equation to the second gives us $-A = -2$ $\Leftrightarrow$ $A = 2$, and hence, $B = -1$.Thus,

$$\int_0^1 \frac{x-4}{x^2-5x+6}\,dx = \int_0^1 \left(\frac{2}{x-2} - \frac{1}{x-3}\right)dx = \left[2\ln|x-2| - \ln|x-3|\right]_0^1$$

$$= (0 - \ln 2) - (2\ln 2 - \ln 3) = -3\ln 2 + \ln 3 \;\; [\text{or } \ln\tfrac{3}{8}]$$

Another method: Substituting 3 for x in the equation $x - 4 = A(x-3) + B(x-2)$ gives $-1 = B$. Substituting 2 for x

gives $-2 = -A$ $\Leftrightarrow$ $A = 2$.

24. $\dfrac{x^2+2x-1}{x^3-x} = \dfrac{x^2+2x-1}{x(x+1)(x-1)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1}$. Multiply both sides by $x(x+1)(x-1)$ to get

$x^2 + 2x - 1 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$ $\Rightarrow$

$x^2 + 2x - 1 = Ax^2 - A + Bx^2 - Bx + Cx^2 + Cx$ $\Rightarrow$

$x^2 + 2x - 1 = (A+B+C)x^2 + (-B+C)x - A$. Equating constant terms, we get $-A = -1$ $\Leftrightarrow$ $A = 1$.

Equating coefficients of x^2 gives $1 = 1 + B + C$ $\Leftrightarrow$ $0 = B + C$. Equating coefficients of x gives $2 = -B + C$.

Adding these equations gives $2 = 2C$ $\Leftrightarrow$ $C = 1$, and hence, $B = -1$. Thus,

$$\int \frac{x^2+2x-1}{x^3-x}\,dx = \int \left(\frac{1}{x} - \frac{1}{x+1} + \frac{1}{x-1}\right)dx = \ln|x| - \ln|x+1| + \ln|x-1| + C = \ln\left|\frac{x(x-1)}{x+1}\right| + C.$$

Another method: Substituting 0 for x in the equation $x^2 + 2x - 1 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$

gives $-1 = -A$ $\Leftrightarrow$ $A = 1$. Substituting -1 for x gives $-2 = 2B$ $\Leftrightarrow$ $B = -1$. Substituting 1 for x gives

$2 = 2C$ $\Leftrightarrow$ $C = 1$.

28. $\dfrac{x^2-x+6}{x^3+3x} = \dfrac{x^2-x+6}{x(x^2+3)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+3}$. Multiply by $x(x^2+3)$ to get $x^2 - x + 6 = A(x^2+3) + (Bx+C)x$.

Substituting 0 for x gives $6 = 3A$ $\Leftrightarrow$ $A = 2$. The coefficients of the x^2-terms must be equal, so $1 = A + B$ $\Rightarrow$

$B = 1 - 2 = -1$. The coefficients of the x-terms must be equal, so $-1 = C$. Thus,

$$\int \frac{x^2-x+6}{x^3+3x}\,dx = \int \left(\frac{2}{x} + \frac{-x-1}{x^2+3}\right)dx = \int \left(\frac{2}{x} - \frac{x}{x^2+3} - \frac{1}{x^2+3}\right)dx$$

$$= 2\ln|x| - \frac{1}{2}\ln(x^2+3) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C$$

Section 5.10

2. (a) Since $y = \dfrac{1}{2x-1}$ is defined and continuous on $[1, 2]$, $\displaystyle\int_1^2 \frac{1}{2x-1}\,dx$ is proper.

(b) Since $y = \dfrac{1}{2x-1}$ has an infinite discontinuity at $x = \frac{1}{2}$, $\displaystyle\int_0^1 \frac{1}{2x-1}\,dx$ is a Type II improper integral.

(c) Since $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

(d) Since $y = \ln(x-1)$ has an infinite discontinuity at $x = 1$, $\int_1^2 \ln(x-1)dx$ is a Type II improper integral.

5. $$\int_3^{\infty} \frac{1}{(x-2)^{3/2}}\,dx = \lim_{t\to\infty} \int_3^t (x-2)^{-3/2}\,dx = \lim_{t\to\infty} \left[-2\,(x-2)^{-1/2}\right]_3^t \qquad [u = x-2,\ du = dx]$$
$$= \lim_{t\to\infty} \left(\frac{-2}{\sqrt{t-2}} + \frac{2}{\sqrt{1}}\right) = 0 + 2 = 2. \qquad \text{Convergent}$$

8. $$\int_0^{\infty} \frac{x}{(x^2+2)^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{x}{(x^2+2)^2}\,dx = \lim_{t\to\infty} \frac{1}{2}\left[\frac{-1}{x^2+2}\right]_0^t = \frac{1}{2}\lim_{t\to\infty}\left(\frac{-1}{t^2+2} + \frac{1}{2}\right)$$
$$= \tfrac{1}{2}\left(0 + \tfrac{1}{2}\right) = \tfrac{1}{4}. \qquad \text{Convergent}$$

10. $\int_{-\infty}^{-1} e^{-2t}\,dt = \lim_{x\to-\infty} \int_x^{-1} e^{-2t}\,dt = \lim_{x\to-\infty} \left[-\frac{1}{2}e^{-2t}\right]_x^{-1} = \lim_{x\to-\infty} \left[-\frac{1}{2}e^2 + \frac{1}{2}e^{-2x}\right] = \infty.$ Divergent

11. $\int_{2\pi}^{\infty} \sin\theta\,d\theta = \lim_{t\to\infty} \int_{2\pi}^{t} \sin\theta\,d\theta = \lim_{t\to\infty} \left[-\cos\theta\right]_{2\pi}^{t} = \lim_{t\to\infty} \left(-\cos t + 1\right)$. This limit does not exist, so the integral is divergent. Divergent

12. $I = \int_{-\infty}^{\infty} (y^3 - 3y^2)\,dy = I_1 + I_2 = \int_{-\infty}^{0} (y^3 - 3y^2)\,dy + \int_0^{\infty} (y^3 - 3y^2)\,dy$, but
$I_1 = \lim_{t\to-\infty} \left[\frac{1}{4}y^4 - y^3\right]_t^0 = \lim_{t\to-\infty} \left(t^3 - \frac{1}{4}t^4\right) = -\infty$. Since I_1 is divergent, I is divergent,
and there is no need to evaluate I_2. Divergent

14. $$\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{t\to\infty} \int_1^t \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{t\to\infty} \int_1^{\sqrt{t}} e^{-u}\,(2\,du) \qquad \left[\begin{array}{l} u = \sqrt{x}, \\ du = dx/(2\sqrt{x}) \end{array}\right]$$
$$= 2\lim_{t\to\infty} \left[-e^{-u}\right]_1^{\sqrt{t}} = 2\lim_{t\to\infty} \left(-e^{-\sqrt{t}} + e^{-1}\right) = 2(0 + e^{-1}) = 2e^{-1}. \qquad \text{Convergent}$$

19. $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^t$ $\left[\begin{array}{l}\text{by substitution with}\\ u=\ln x,\, du=dx/x\end{array}\right]$ $\displaystyle= \lim_{t\to\infty}\frac{(\ln t)^2}{2} = \infty.$ Divergent

20. $I = \int_{-\infty}^{\infty} x^3 e^{-x^4}\,dx = I_1 + I_2 = \int_{-\infty}^{0} x^3 e^{-x^4}\,dx + \int_0^\infty x^3 e^{-x^4}\,dx$. Now

$$I_2 = \lim_{t\to\infty}\int_0^t x^3 e^{-x^4}\,dx = \lim_{t\to\infty}\int_0^{t^4} e^{-u}\left(\tfrac{1}{4}\,du\right) \quad \left[\begin{array}{l}u=x^4,\\ du=4x^3\,dx\end{array}\right]$$

$$= \tfrac{1}{4}\lim_{t\to\infty}\left[-e^{-u}\right]_0^{t^4} = \tfrac{1}{4}\lim_{t\to\infty}\left(-e^{-t^4}+1\right) = \tfrac{1}{4}(0+1) = \tfrac{1}{4}.$$

Since $f(x) = x^3 e^{-x^4}$ is an odd function, $I_1 = -\frac{1}{4}$, and hence, $I = 0$. Convergent

25. $\displaystyle\int_0^1 \frac{3}{x^5}\,dx = \lim_{t\to0^+}\int_t^1 3x^{-5}\,dx = \lim_{t\to0^+}\left[-\frac{3}{4x^4}\right]_t^1 = -\frac{3}{4}\lim_{t\to0^+}\left(1-\frac{1}{t^4}\right) = \infty.$ Divergent

30. $f(y) = 1/(4y-1)$ has an infinite discontinuity at $y = \frac{1}{4}$.

$$\int_{1/4}^1 \frac{1}{4y-1}\,dy = \lim_{t\to(1/4)^+}\int_t^1 \frac{1}{4y-1}\,dy = \lim_{t\to(1/4)^+}\left[\tfrac{1}{4}\ln|4y-1|\right]_t^1 = \lim_{t\to(1/4)^+}\left[\tfrac{1}{4}\ln 3 - \tfrac{1}{4}\ln(4t-1)\right] = \infty,$$

so $\displaystyle\int_{1/4}^1 \frac{1}{4y-1}\,dy$ diverges, and hence, $\displaystyle\int_0^1 \frac{1}{4y-1}\,dy$ diverges. Divergent

33. $I = \int_0^2 z^2 \ln z\,dz = \lim_{t\to0^+}\int_t^2 z^2\ln z\,dz = \lim_{t\to0^+}\left[\frac{z^3}{3^2}(3\ln z - 1)\right]_t^2$ $\left[\begin{array}{l}\text{integrate by parts}\\ \text{or use Formula 101}\end{array}\right]$

$= \lim_{t\to0^+}\left[\frac{8}{9}(3\ln 2 - 1) - \frac{1}{9}t^3(3\ln t - 1)\right] = \frac{8}{3}\ln 2 - \frac{8}{9} - \frac{1}{9}\lim_{t\to0^+}\left[t^3(3\ln t-1)\right] = \frac{8}{3}\ln 2 - \frac{8}{9} - \frac{1}{9}L.$

Now $L = \displaystyle\lim_{t\to0^+}\left[t^3(3\ln t - 1)\right] = \lim_{t\to0^+}\frac{3\ln t - 1}{t^{-3}} \overset{\text{H}}{=} \lim_{t\to0^+}\frac{3/t}{-3/t^4} = \lim_{t\to0^+}(-t^3) = 0.$

Thus, $L = 0$ and $I = \frac{8}{3}\ln 2 - \frac{8}{9}$. Convergent

49. $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \int_0^1 \frac{dx}{\sqrt{x}\,(1+x)} + \int_1^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \lim_{t\to0^+}\int_t^1 \frac{dx}{\sqrt{x}\,(1+x)} + \lim_{t\to\infty}\int_1^t \frac{dx}{\sqrt{x}\,(1+x)}.$ Now

$\displaystyle\int \frac{dx}{\sqrt{x}\,(1+x)} = \int \frac{2u\,du}{u(1+u^2)}$ $\left[\begin{array}{l}u=\sqrt{x},\, x=u^2,\\ dx = 2u\,du\end{array}\right]$ $\displaystyle= 2\int\frac{du}{1+u^2} = 2\tan^{-1}u + C = 2\tan^{-1}\sqrt{x} + C,$ so

$$\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \lim_{t\to0^+}\left[2\tan^{-1}\sqrt{x}\right]_t^1 + \lim_{t\to\infty}\left[2\tan^{-1}\sqrt{x}\right]_1^t$$

$$= \lim_{t\to0^+}\left[2\left(\tfrac{\pi}{4}\right) - 2\tan^{-1}\sqrt{t}\right] + \lim_{t\to\infty}\left[2\tan^{-1}\sqrt{t} - 2\left(\tfrac{\pi}{4}\right)\right] = \tfrac{\pi}{2} - 0 + 2\left(\tfrac{\pi}{2}\right) - \tfrac{\pi}{2} = \pi.$$

61. We use integration by parts: let $u = x$, $dv = xe^{-x^2}\,dx \;\Rightarrow\; du = dx$, $v = -\frac{1}{2}e^{-x^2}$. So

$$\int_0^\infty x^2 e^{-x^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2}xe^{-x^2}\right]_0^t + \frac{1}{2}\int_0^\infty e^{-x^2}\,dx = \lim_{t\to\infty}\left[-\frac{t}{2e^{t^2}}\right] + \frac{1}{2}\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty e^{-x^2}\,dx$$

(The limit is 0 by l'Hospital's Rule.)

Section 6.1

1. $A = \int_{x=0}^{x=4} (y_T - y_B)\,dx = \int_0^4 \left[(5x - x^2) - x\right] dx = \int_0^4 (4x - x^2)\,dx = \left[2x^2 - \frac{1}{3}x^3\right]_0^4 = \left(32 - \frac{64}{3}\right) - (0) = \frac{32}{3}$

2. $A = \int_0^2 \left(\sqrt{x+2} - \frac{1}{x+1}\right) dx = \left[\frac{2}{3}(x+2)^{3/2} - \ln(x+1)\right]_0^2$

$= \left[\frac{2}{3}(4)^{3/2} - \ln 3\right] - \left[\frac{2}{3}(2)^{3/2} - \ln 1\right] = \frac{16}{3} - \ln 3 - \frac{4}{3}\sqrt{2}$

13. $12 - x^2 = x^2 - 6 \quad \Leftrightarrow \quad 2x^2 = 18 \quad \Leftrightarrow$

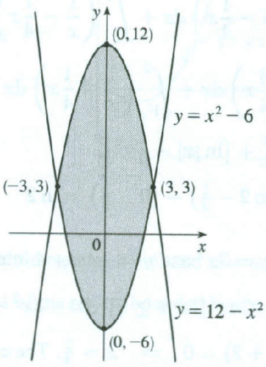
$x^2 = 9 \quad \Leftrightarrow \quad x = \pm 3$, so

$A = \int_{-3}^{3} \left[(12 - x^2) - (x^2 - 6)\right] dx$

$= 2\int_0^3 (18 - 2x^2)\,dx$ [by symmetry]

$= 2\left[18x - \frac{2}{3}x^3\right]_0^3 = 2\left[(54 - 18) - 0\right]$

$= 2(36) = 72$



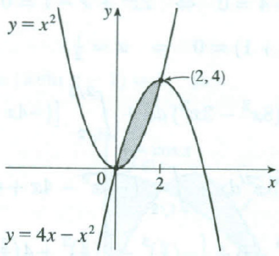
14. $x^2 = 4x - x^2 \quad \Leftrightarrow \quad 2x^2 - 4x = 0 \quad \Leftrightarrow \quad 2x(x-2) = 0 \quad \Leftrightarrow \quad x = 0$ or 2, so

$A = \int_0^2 \left[(4x - x^2) - x^2\right] dx$

$= \int_0^2 (4x - 2x^2)\,dx$

$= \left[2x^2 - \frac{2}{3}x^3\right]_0^2$

$= 8 - \frac{16}{3} = \frac{8}{3}$

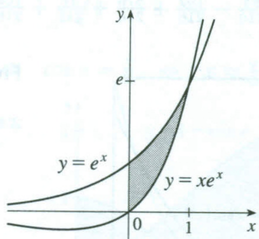


15. $e^x = xe^x \quad \Leftrightarrow \quad e^x - xe^x = 0 \quad \Leftrightarrow \quad e^x(1-x) = 0 \quad \Leftrightarrow \quad x = 1.$

$A = \int_0^1 (e^x - xe^x)\,dx$

$= \left[e^x - (xe^x - e^x)\right]_0^1$ [use parts with $u = x$ and $dv = e^x\,dx$]

$= \left[2e^x - xe^x\right]_0^1 = (2e - e) - (2 - 0) = e - 2$

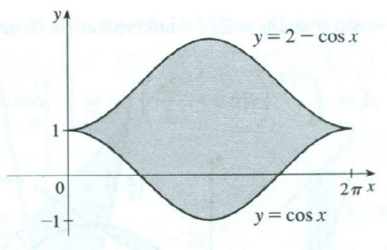


16. $A = \int_0^{2\pi} \left[(2 - \cos x) - \cos x\right] dx$

$= \int_0^{2\pi} (2 - 2\cos x)\,dx$

$= \left[2x - 2\sin x\right]_0^{2\pi}$

$= (4\pi - 0) - 0 = 4\pi$



24.

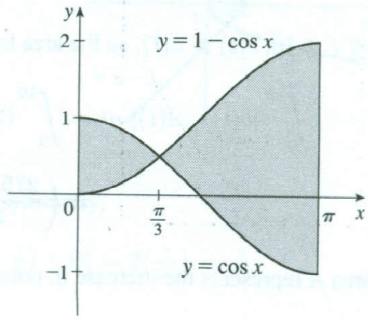
The curves intersect when $\cos x = 1 - \cos x$ (on $[0, \pi]$) $\Leftrightarrow$ $2\cos x = 1$ $\Leftrightarrow$ $\cos x = \frac{1}{2}$ $\Leftrightarrow$ $x = \frac{\pi}{3}$.

$$A = \int_0^{\pi/3} [\cos x - (1 - \cos x)]\, dx + \int_{\pi/3}^{\pi} [(1 - \cos x) - \cos x]\, dx$$

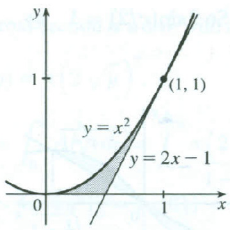
$$= \int_0^{\pi/3} (2\cos x - 1)\, dx + \int_{\pi/3}^{\pi} (1 - 2\cos x)\, dx$$

$$= \Big[2\sin x - x\Big]_0^{\pi/3} + \Big[x - 2\sin x\Big]_{\pi/3}^{\pi}$$

$$= \left(\sqrt{3} - \frac{\pi}{3}\right) - 0 + (\pi - 0) - \left(\frac{\pi}{3} - \sqrt{3}\right) = 2\sqrt{3} + \frac{\pi}{3}$$



40.



We start by finding the equation of the tangent line to $y = x^2$ at the point $(1, 1)$: $y' = 2x$, so the slope of the tangent is $2(1) = 2$, and its equation is $y - 1 = 2(x - 1)$, or $y = 2x - 1$. We would need two integrals to integrate with respect to x, but only one to integrate with respect to y.

$$A = \int_0^1 \left[\tfrac{1}{2}(y + 1) - \sqrt{y}\right] dy = \left[\tfrac{1}{4}y^2 + \tfrac{1}{2}y - \tfrac{2}{3}y^{3/2}\right]_0^1$$

$$= \tfrac{1}{4} + \tfrac{1}{2} - \tfrac{2}{3} = \tfrac{1}{12}$$