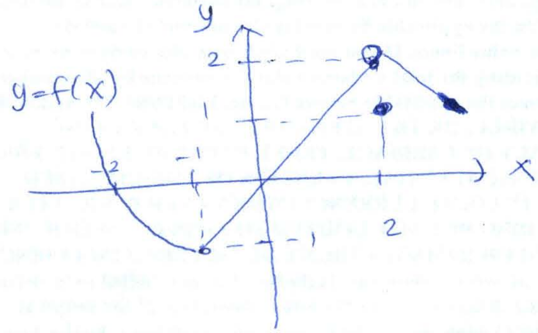
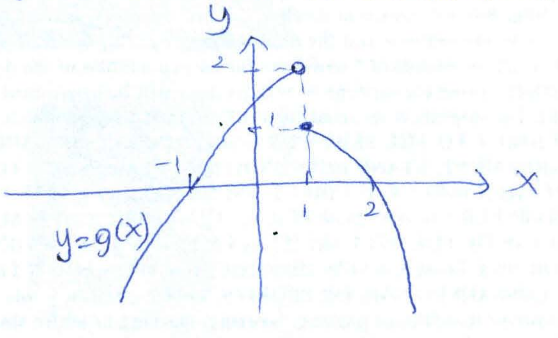
Homework #3

Section 2.3: Calculating limits using the Limit Laws

2) The graphs of f and g are given. Use them to evaluate the limit, if it exists.





(a) $\lim\limits_{x\to 2} [f(x)+g(x)] = \lim\limits_{x\to 2} f(x) + \lim\limits_{x\to 2} g(x) = 2+0=2$

(b) $\lim\limits_{x\to 1} [f(x)+g(x)]$; $\lim\limits_{x\to 1} g(x)$ doesn't exist. So the given limit doesn't exist.

(c) $\lim\limits_{x\to 0} [f(x)\,g(x)] = \lim\limits_{x\to 0} f(x) \,.\, \lim\limits_{x\to 0} g(x) = 0.(1.3) = 0$.

(d) $\lim\limits_{x+1} \dfrac{f(x)}{g(x)}$; Since $\lim\limits_{x\to -1} g(x)=0$. and g is in the denominator, the given limit doesn't exist.

(e) $\lim\limits_{x\to 2} (x^3 f(x)) = (\lim\limits_{x\to 2} x^3)(\lim\limits_{x\to 2} f(x)) = 2^3 . 2 = 16$

(f) $\lim\limits_{x\to 1} \sqrt{3+f(x)} = \sqrt{3+\lim\limits_{x\to 1} f(x)} = \sqrt{3+1} = 2$

6) $\lim\limits_{u\to -2} \sqrt{u^4+3u+6} = \sqrt{\lim\limits_{u\to -2} (u^4+3u+6)} = \sqrt{\lim\limits_{u\to -2} u^4 + 3\lim\limits_{u\to -2} u + \lim\limits_{u\to -2} 6}$

$= \sqrt{(-2)^4+3(-2)+6}$

$= \sqrt{16-6+6} = 4.$

7) $\lim\limits_{x\to 2} \sqrt{\dfrac{2x^2+1}{3x-2}} = \dfrac{\sqrt{\lim\limits_{x\to 2} 2x^2+1}}{\sqrt{\lim\limits_{x\to 2} 3x-2}} = \dfrac{\sqrt{2\lim\limits_{x\to 2} x^2+1}}{\sqrt{3\lim\limits_{x\to 2} x-2}} = \dfrac{\sqrt{2.4+1}}{\sqrt{3.2-2}} = \dfrac{3}{2} = 1.5$

10) $\lim_{x\to -4} \dfrac{x^2+5x+4}{x^2+3x-4} = \lim_{x\to -4} \dfrac{\cancel{(x+4)}(x+1)}{\cancel{(x+4)}(x-1)} = \lim_{x\to -4} \dfrac{x+1}{x-1} = \dfrac{\lim_{x\to -4}(x+1)}{\lim_{x\to -4}(x-1)} = \dfrac{-3}{-5}$

$= 3/5$

13) $\lim_{t\to -3} \dfrac{t^2-9}{2t^2+7t+3} = \lim_{t\to -3} \dfrac{\cancel{(t+3)}(t-3)}{(2t+1)\cancel{(t+3)}} = \lim_{t\to -3} \dfrac{t-3}{2t+1} = \dfrac{-3-3}{2(-3)+1} = \dfrac{-6}{-5} = \dfrac{6}{5}$

17) $\lim_{x\to -2} \dfrac{x+2}{x^3+8} = \lim_{x\to -2} \dfrac{x+2}{(x+2)(x^2-2x+4)} = \lim_{x\to -2} \dfrac{1}{x^2-2x+4} = \dfrac{1}{4+4+4} = 1/12$

21) $\lim_{x\to 16} \dfrac{4-\sqrt{x}}{16x-x^2} = \lim_{x\to 16} \dfrac{4-\sqrt{x}}{x(16-x)} = \lim_{x\to 16} \dfrac{\cancel{4-\sqrt{x}}}{x\cancel{(4-\sqrt{x})}(4+\sqrt{x})}$

$= \lim_{x\to 16} \dfrac{1}{x(4+\sqrt{x})} = \dfrac{1}{16(4+\sqrt{16})} = 1/128$

24) $\lim_{x\to -4} \dfrac{\sqrt{x^2+9}-5}{x+4} = \lim_{x\to -4} \dfrac{(\sqrt{x^2+9}-5)(\sqrt{x^2+9}+5)}{(x+4)(\sqrt{x^2+9}+5)}$

$= \lim_{x\to -4} \dfrac{x^2+9-25}{(x+4)(\sqrt{x^2+9}+5)} = \lim_{x\to -4} \dfrac{x^2-16}{(x+4)(\sqrt{x^2+9}+5)}$

$= \lim_{x\to -4} \dfrac{\cancel{(x+4)}(x-4)}{\cancel{(x+4)}(\sqrt{x^2+9}+5)} = \lim_{x\to -4} \dfrac{(x-4)}{\sqrt{x^2+9}+5} = \dfrac{-4-4}{\sqrt{16+9}+5} = \dfrac{-8}{10}$

30) If $2x \leq g(x) \leq x^4-x^2+2$ for all x, evaluate $\lim_{x\to 1} g(x)$

$\lim_{x\to 1}(2x) = 2(1) = 2$ and $\lim_{x\to 1}(x^4-x^2+2) = 1^4-1^2+2 = 2.$

Since $2x \leq g(x) \leq x^4-x^2+2$ for all x,

$\lim_{x\to 1} g(x) = 2$ by the Squeeze Thm.

31) Prove that $\lim_{x\to 0^+} x^4 \cos\frac{2}{x} = 0$.

$-1 \leq \cos\frac{2}{x} \leq 1 \Rightarrow -x^4 \leq x^4\cos\frac{2}{x} \leq x^4$

Since $\lim_{x\to 0}(-x^4) = \lim_{x\to 0}(x4) = 0$. we have $\lim_{x\to 0}\left(x4\cos\left(\frac{2}{x}\right)\right) = 0$ by squeeze Thm.

34) $\lim_{x\to -6} \frac{2x+12}{|x+6|} = ?$

$$|x+6| = \begin{cases} x+6 & \text{if } x+6 \geq 0 \\ -(x+6) & \text{if } x+6<0 \end{cases} = \begin{cases} x+6, & x\geq -6 \\ -(x+6), & x<-6 \end{cases}$$

Check one-sided limits,

$\lim_{x\to -6^+} \frac{2x+12}{|x+6|} = \lim_{x\to -6^+} \frac{2(x+6)}{x+6} = 2$

$\lim_{x\to -6^-} \frac{2x+12}{-(x+6)} = \lim_{x\to -6^+} \frac{2\cancel{(x+6)}}{-\cancel{(x+6)}} = -2.$

limits are different, so $\lim_{x\to -6} \frac{2x+12}{|x+6|}$ doesn't exist.

36) $\lim_{x\to -2} \frac{2-|x|}{2+x} = ?$ $|x| = \begin{cases} x, & x\geq 0 \\ -x, & x<0 \end{cases}$ hence, $x=-2$ isn't a critical point.

$\lim_{x\to -2^+} \frac{2-|x|}{2+x} = \lim_{x\to -2^+} \frac{2+x}{2+x} = 1$. and $\lim_{x\to -2^-} \frac{2-|x|}{2+x} = \lim_{x\to -2^+} \frac{2+x}{2+x} = 1$

$\lim_{x\to -2} \frac{2-|x|}{2+x} = 1.$

37) $$g(x) = \begin{cases} x & \text{if } x<1 \\ 3 & \text{if } x=1 \\ 2-x^2 & \text{if } 1<x\leq 2 \\ x-3 & \text{if } x>2 \end{cases}$$

(i) $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} x = 1.$

(ii) $\lim_{x\to 1} g(x) = 1$ $\Leftarrow$ equal

$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+}(2-x^2) = 2-1 = 1$

(iii) $g(1) = 3.$

(iv) $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-}(2-x^2) = -2$

(v) $\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+}(x-3) = -1$

Different

(vi) $\lim_{x\to 2} g(x)$ doesn't exist.

49) Is there a number a such that

$$\lim_{x\to -2} \frac{3x^2+ax+a+3}{x^2+x-2}$$

exists? If so, find the value of a and the limit.

Since the denominator approaches 0 as $x\to -2$, the limit will exist only if the numerator also approaches 0 as $x\to-2$. Hence we need,

$$\lim_{x\to -2} (3x^2+ax+a+3)=0 .$$

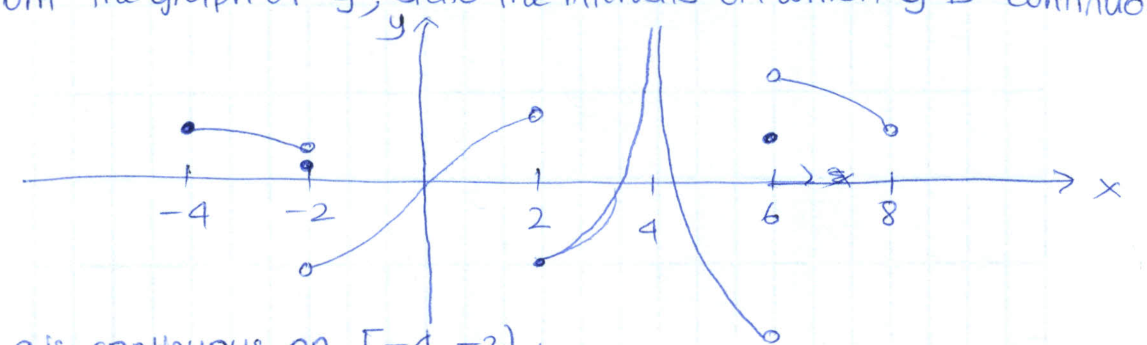
$$3.(-2)^2+a(-2)+a+3=0 \iff 12-2a+a+3=0 \iff a=15$$

With $a=15$,

$$\lim_{x\to -2} \frac{3x^2+15x+18}{x^2+x-2} = \lim_{x\to -2} \frac{3\cancel{(x+2)}(x+3)}{(x-1)\cancel{(x+2)}} = \lim_{x\to -2} \frac{3(x+3)}{(x-1)} = \frac{3}{-3} = -1.$$

Section 2.4: Continuity

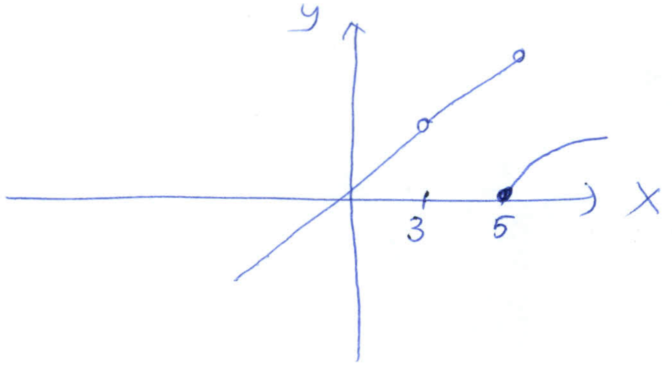
4) From the graph of g, state the intervals on which g is continuous.



g is continuous on $[-4,-2)$,

$(-2,2)$, $[2,4)$, $(4,6)$, $(6,8)$

7) Sketch the graph of "Removable discontinuity at 3, jump discontinuity at 5"



11) If f and g are continuous functions with $f(3)=5$ and $\lim_{x\to3}(2f(x)-g(x))=4$, find $g(3)$.

If f and g are continuous then, $\lim_{x\to3} f(x) = f(3)$ and $\lim_{x\to3} g(x) = g(3)$

$\lim_{x\to3}(2f(x)-g(x)) = 2\lim_{x\to3} f(x) - \lim_{x\to3} g(x) = 2.5 - g(3) = 4 \Rightarrow$ ~~g(3)=14~~
g(3) = 6

15) $f(x) = \begin{cases} e^x, & \text{if } x<0 \\ x^2, & \text{if } x\geq 0 \end{cases}$ $(a=0)$

The left-hand limit of f at $a=0$ is $\lim_{x\to0^-} f(x) = \lim_{x\to0^-} e^x = 1$.
The right " " " " " $\lim_{x\to0^+} f(x) = \lim_{x\to0^+} x^2 = 0$.

Since these aren't equal, $\lim_{x\to0} f(x)$ doesn't exist, f is discontinuous.

16) $f(x) = \begin{cases} \dfrac{x^2-x}{x^2-1} & \text{if } x\neq 1 \\ 1 & \text{if } x=1 \end{cases}$ $(a=1)$

$\lim_{x\to1} f(x) = \lim_{x\to1} \dfrac{x^2-x}{x^2-1} = \lim_{x\to1} \dfrac{x(x-1)}{(x+1)(x-1)} = \lim_{x\to1} \dfrac{x}{x+1} = \dfrac{1}{2}$

but $f(1)=1$. So, f is discontinuous at 1.

18) $f(x) = \begin{cases} \dfrac{2x^2-5x-3}{x-3} & \text{if } x\neq 3 \\ 6 & \text{if } x=3 \end{cases}$ $(a=3)$

$\lim_{x\to3} f(x) = \lim_{x\to3} \dfrac{2x^2-5x-3}{x-3} = \lim_{x\to3} \dfrac{(2x+1)\cancel{(x-3)}}{\cancel{x-3}} = 7$

but $f(3)=6$. So, f is discontinuous at 3.

23) $G(t)=\ln(t^4-1)$

We know from Thm 7 that the function $y=\ln x$ is continuous for $x>0$.

The domain of $G(t)$ is, $t^4-1>0 \Rightarrow t^4>1 \Rightarrow t\in(-\infty,-1)\cup(1,\infty)$

$G(t)$ is continuous on its domain

28) $\lim\limits_{x\to\pi} \sin(x+\sin x)=?$

Because x is continuous on $\mathbb{R}$, $\sin x$ is continuous on $\mathbb{R}$ and $x+\sin x$ is continuous on $\mathbb{R}$, the composite function $f(x)=\sin(x+\sin x)$ is cont. on $\mathbb{R}$.

So, $\lim\limits_{x\to\pi} f(x)=f(\pi)=\sin(\pi+\sin\pi)=\sin\pi=0$.

29) $\lim\limits_{x\to 1} e^{x^2-x}=?$ Because x^2-x is cont. on $\mathbb{R}$, the composite function $f(x)=e^{x^2-x}$ is continuous on $\mathbb{R}$, so $\lim\limits_{x\to 1} f(x)=f(1)=e^{1-1}=e^0=1$.

33) $f(x)=\begin{cases} x+2 & \text{if } x<0 \\ e^x & \text{if } 0\le x\le 1 \\ 2-x & \text{if } x>1 \end{cases}$

f is continuous on $(-\infty,0)$ and $(1,\infty)$ since on each of these intervals it is a polynomial, it is continuous on $(0,1)$ since it is an exponential.

Now $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}(x+2)=2$ and $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+} e^x=1$, so f is discontinuous at 0.

f is cont. from right at 0. Also, $\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-} e^x=e$ and $\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}(2-x)=1$

f is discontinuous at 1. Since $f(1)=e$, f is cont. from left at 1.

35) For what value of the constant c is the function f continuous on $(-\infty,\infty)$?

$$f(x)=\begin{cases} cx^2+2x & \text{if } x<2 \\ x^3-cx & \text{if } x\geq 2 \end{cases}$$

f is continous on $(-\infty,2)$ and $(2,\infty)$. Now $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(cx^2+2x)=4c+4$

and $\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(x^3-cx)=8-2c$, So f is conti $\Leftrightarrow$ $4c+4=8-2c$

$6c=4 \Rightarrow c=2/3$.

45) a) Prove that $\cos x=x^3$ has at least one real root.

$f(x)=\cos x-x^3$ is cont. on $[0,1]$. $f(0)=1>0$ $f(1)=\cos 1-1=-0.46<0$

Since $1>0>-0.46$, there's a number c in $(0,1)$ such that $f(c)=0$. (Intermediate Value Thm.)

b) $f(0.86)\approx 0.016>0$ and $f(0.87)\approx -0.014<0$. so there's one root between 0.86 and 0.87.

51) Is there a number that's exactly 1 more than its cube?

$x-x^3=1$

$x^3-x+1=0$.

f(-2) = 6 and f(0) = 0.

Using the Intermediate Value Theorem, f(x) = 1 for some x € (-2 , 0).