Homework 4 – Solutions

Section 2.5: Limits involving infinity

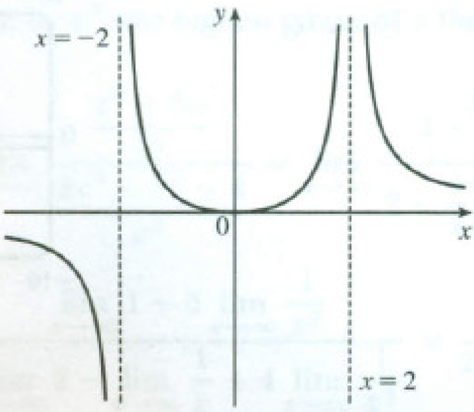
4) (a) $\lim_{x \to \infty} g(x) = 2$ (b) $\lim_{x \to -\infty} g(x) = -2$ (c) $\lim_{x \to 3} g(x) = \infty$

(d) $\lim_{x \to 0} g(x) = -\infty$ (e) $\lim_{x \to -2^+} g(x) = -\infty$ (f) Vertical: $x = -2, x = 0, x = 3$; Horizontal: $y = -2, y = 2$

6. $\lim_{x \to 2} f(x) = \infty, \quad \lim_{x \to -2^+} f(x) = \infty,$

$\lim_{x \to -2^-} f(x) = -\infty, \quad \lim_{x \to -\infty} f(x) = 0,$

$\lim_{x \to \infty} f(x) = 0, \quad f(0) = 0$



15) $\lim_{x \to 1} \frac{2 - x}{(x - 1)^2} = \infty$ since the numerator is positive and the denominator approaches 0 through positive values as $x \to 1$.

17) Let $t = 3/(2 - x)$. As $x \to 2^+$, $t \to -\infty$. So $\lim_{x \to 2^+} e^{3/(2-x)} = \lim_{t \to -\infty} e^t = 0$ by (7).

18) $\lim_{x \to \pi^-} \cot x = \lim_{x \to \pi^-} \frac{\cos x}{\sin x} = -\infty$ since the numerator is negative and the denominator approaches 0 through positive values as $x \to \pi^-$.

20) $\lim_{x \to 2^-} \frac{x^2 - 2x}{x^2 - 4x + 4} = \lim_{x \to 2^-} \frac{x(x - 2)}{(x - 2)^2} = \lim_{x \to 2^-} \frac{x}{x - 2} = -\infty$ since the numerator is positive and the denominator approaches 0 through negative values as $x \to 2^-$.

22) $\lim_{x\to\infty} \frac{3x+5}{x-4} = \lim_{x\to\infty} \frac{(3x+5)/x}{(x-4)/x} = \lim_{x\to\infty} \frac{3+5/x}{1-4/x} = \frac{\lim_{x\to\infty} 3+5 \lim_{x\to\infty} \frac{1}{x}}{\lim_{x\to\infty} 1-4 \lim_{x\to\infty} \frac{1}{x}} = \frac{3+5(0)}{1-4(0)} = 3$

23) Divide both the numerator and denominator by x^3 (the highest power of x that occurs in the denominator).

$$\lim_{x\to\infty} \frac{x^3+5x}{2x^3-x^2+4} = \lim_{x\to\infty} \frac{\frac{x^3+5x}{x^3}}{\frac{2x^3-x^2+4}{x^3}} = \lim_{x\to\infty} \frac{1+\frac{5}{x^2}}{2-\frac{1}{x}+\frac{4}{x^3}} = \frac{\lim_{x\to\infty}\left(1+\frac{5}{x^2}\right)}{\lim_{x\to\infty}\left(2-\frac{1}{x}+\frac{4}{x^3}\right)}$$

$$= \frac{\lim_{x\to\infty} 1+5\lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty} 2-\lim_{x\to\infty}\frac{1}{x}+4\lim_{x\to\infty}\frac{1}{x^3}} = \frac{1+5(0)}{2-0+4(0)} = \frac{1}{2}$$

26. $\lim_{x\to\infty} \frac{x+2}{\sqrt{9x^2+1}} = \lim_{x\to\infty} \frac{(x+2)/x}{\sqrt{9x^2+1}/\sqrt{x^2}} = \lim_{x\to\infty} \frac{1+2/x}{\sqrt{9+1/x^2}} = \frac{1+0}{\sqrt{9+0}} = \frac{1}{3}$

27. $\lim_{x\to\infty} \left(\sqrt{9x^2+x}-3x\right) = \lim_{x\to\infty} \frac{\left(\sqrt{9x^2+x}-3x\right)\left(\sqrt{9x^2+x}+3x\right)}{\sqrt{9x^2+x}+3x} = \lim_{x\to\infty} \frac{\left(\sqrt{9x^2+x}\right)^2-(3x)^2}{\sqrt{9x^2+x}+3x}$

$$= \lim_{x\to\infty} \frac{(9x^2+x)-9x^2}{\sqrt{9x^2+x}+3x} = \lim_{x\to\infty} \frac{x}{\sqrt{9x^2+x}+3x} \cdot \frac{1/x}{1/x}$$

$$= \lim_{x\to\infty} \frac{x/x}{\sqrt{9x^2/x^2+x/x^2}+3x/x} = \lim_{x\to\infty} \frac{1}{\sqrt{9+1/x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{3+3} = \frac{1}{6}$$

29. Let $t=-x^2$. As $x\to\infty$, $t\to-\infty$. So $\lim_{x\to\infty} e^{-x^2} = \lim_{t\to-\infty} e^t = 0$ by (7).

30. For $x>0$, $\sqrt{x^2+1} > \sqrt{x^2} = x$. So as $x\to\infty$, we have $\sqrt{x^2+1}\to\infty$, that is, $\lim_{x\to\infty}\sqrt{x^2+1}=\infty$.

33. Since $-1\le \cos x\le 1$ and $e^{-2x}>0$, we have $-e^{-2x}\le e^{-2x}\cos x\le e^{-2x}$. We know that $\lim_{x\to\infty}(-e^{-2x})=0$ and $\lim_{x\to\infty}(e^{-2x})=0$, so by the Squeeze Theorem, $\lim_{x\to\infty}(e^{-2x}\cos x)=0$.

34. Divide numerator and denominator by e^{3x}: $\lim_{x\to\infty} \frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}} = \lim_{x\to\infty} \frac{1-e^{-6x}}{1+e^{-6x}} = \frac{1-0}{1+0} = 1$

37. $\lim_{x\to\infty} \frac{x+x^3+x^5}{1-x^2+x^4} = \lim_{x\to\infty} \frac{(x+x^3+x^5)/x^4}{(1-x^2+x^4)/x^4}$ [divide by the highest power of x in the denominator]

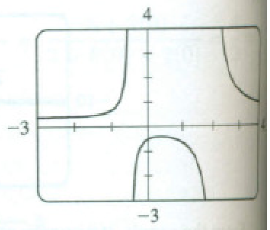
$$= \lim_{x\to\infty} \frac{1/x^3+1/x+x}{1/x^4-1/x^2+1} = \infty$$

because $(1/x^3+1/x+x)\to\infty$ and $(1/x^4-1/x^2+1)\to 1$ as $x\to\infty$.

40. $\displaystyle \lim_{x\to\infty} \frac{x^2+1}{2x^2-3x-2} = \lim_{x\to\infty} \frac{\dfrac{x^2+1}{x^2}}{\dfrac{2x^2-3x-2}{x^2}} = \lim_{x\to\infty} \frac{1+\dfrac{1}{x^2}}{2-\dfrac{3}{x}-\dfrac{2}{x^2}} = \frac{\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x^2}\right)}{\lim\limits_{x\to\infty}\left(2-\dfrac{3}{x}-\dfrac{2}{x^2}\right)}$

$$= \frac{\lim\limits_{x\to\infty} 1 + \lim\limits_{x\to\infty} \dfrac{1}{x^2}}{\lim\limits_{x\to\infty} 2 - \lim\limits_{x\to\infty} \dfrac{3}{x} - \lim\limits_{x\to\infty} \dfrac{2}{x^2}} = \frac{1+0}{2-0-0} = \frac{1}{2},$$ so $y = \frac{1}{2}$ is a horizontal asymptote.

$y = f(x) = \dfrac{x^2+1}{2x^2-3x-2} = \dfrac{x^2+1}{(2x+1)(x-2)}$, so $\lim\limits_{x\to(-1/2)^-} f(x) = \infty$ because as $x \to (-1/2)^-$ the numerator is positive while the denominator approaches 0 through positive values. Similarly, $\lim\limits_{x\to(-1/2)^+} f(x) = -\infty$, $\lim\limits_{x\to 2^-} f(x) = -\infty$, and $\lim\limits_{x\to 2^+} f(x) = \infty$. Thus, $x = -\frac{1}{2}$ and $x = 2$ are vertical asymptotes. The graph confirms our work.



x = 4

47. Let's look for a rational function.

(1) $\lim\limits_{x\to\pm\infty} f(x) = 0 \;\Rightarrow\;$ degree of numerator < degree of denominator

(2) $\lim\limits_{x\to 0} f(x) = -\infty \;\Rightarrow\;$ there is a factor of x^2 in the denominator (not just x, since that would produce a sign change at $x = 0$), and the function is negative near $x = 0$.

(3) $\lim\limits_{x\to 3^-} f(x) = \infty$ and $\lim\limits_{x\to 3^+} f(x) = -\infty \;\Rightarrow\;$ vertical asymptote at $x = 3$; there is a factor of $(x-3)$ in the denominator.

(4) $f(2) = 0 \;\Rightarrow\;$ 2 is an x-intercept; there is at least one factor of $(x-2)$ in the numerator.

Combining all of this information and putting in a negative sign to give us the desired left- and right-hand limits gives us $f(x) = \dfrac{2-x}{x^2(x-3)}$ as one possibility.

Section 2.6: Derivatives and Rates of Change

6. Using (2) with $f(x) = x^3 - 3x + 1$ and $P(2, 3)$,

$$m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{(2+h)^3 - 3(2+h) + 1 - 3}{h}$$
$$= \lim_{h\to 0} \frac{8+12h+6h^2+h^3-6-3h-2}{h} = \lim_{h\to 0} \frac{9h+6h^2+h^3}{h} = \lim_{h\to 0} \frac{h(9+6h+h^2)}{h}$$
$$= \lim_{h\to 0}(9+6h+h^2) = 9$$

Tangent line: $y - 3 = 9(x-2) \;\Leftrightarrow\; y - 3 = 9x - 18 \;\Leftrightarrow\; y = 9x - 15$

7. Using (1), $m = \lim\limits_{x\to 1} \dfrac{\sqrt{x}-\sqrt{1}}{x-1} = \lim\limits_{x\to 1} \dfrac{(\sqrt{x}-1)(\sqrt{x}+1)}{(x-1)(\sqrt{x}+1)} = \lim\limits_{x\to 1} \dfrac{x-1}{(x-1)(\sqrt{x}+1)} = \lim\limits_{x\to 1} \dfrac{1}{\sqrt{x}+1} = \dfrac{1}{2}$.

Tangent line: $y - 1 = \frac{1}{2}(x-1) \;\Leftrightarrow\; y = \frac{1}{2}x + \frac{1}{2}$

8. Using (1) with $f(x) = \dfrac{2x+1}{x+2}$ and $P(1,1)$,

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 1} \frac{\dfrac{2x+1}{x+2} - 1}{x - 1} = \lim_{x \to 1} \frac{\dfrac{2x+1-(x+2)}{x+2}}{x-1} = \lim_{x \to 1} \frac{x-1}{(x-1)(x+2)}$$

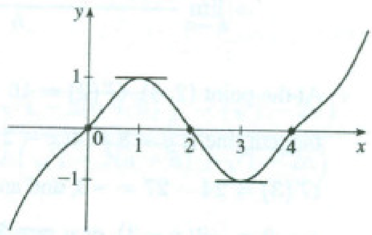
$$= \lim_{x \to 1} \frac{1}{x+2} = \frac{1}{1+2} = \frac{1}{3}$$

Tangent line: $y - 1 = \frac{1}{3}(x - 1) \quad \Leftrightarrow \quad y - 1 = \frac{1}{3}x - \frac{1}{3} \quad \Leftrightarrow \quad y = \frac{1}{3}x + \frac{2}{3}$

17) $g'(0)$ is the only negative value. The slope at $x = 4$ is smaller than the slope at $x = 2$ and both are smaller than the slope at $x = -2$. Thus, $g'(0) < 0 < g'(4) < g'(2) < g'(-2)$.

19) For the tangent line $y = 4x - 5$: when $x = 2$, $y = 4(2) - 5 = 3$ and its slope is 4 (the coefficient of x). At the point of tangency, these values are shared with the curve $y = f(x)$; that is, $f(2) = 3$ and $f'(2) = 4$.

22) We begin by drawing a curve through the origin with a slope of 1 to satisfy $g(0) = 0$ and $g'(0) = 1$. We round off our figure at $x = 1$ to satisfy $g'(1) = 0$, and then pass through $(2, 0)$ with slope -1 to satisfy $g(2) = 0$ and $g'(2) = -1$. We round the figure at $x = 3$ to satisfy $g'(3) = 0$, and then pass through $(4, 0)$ with slope 1 to satisfy $g(4) = 0$ and $g'(4) = 1$. Finally we extend the curve on both ends to satisfy $\lim_{x \to \infty} g(x) = \infty$ and $\lim_{x \to -\infty} g(x) = -\infty$.



23) Using (4) with $f(x) = 3x^2 - x^3$ and $a = 1$,

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{[3(1+h)^2 - (1+h)^3] - 2}{h}$$

$$= \lim_{h \to 0} \frac{(3 + 6h + 3h^2) - (1 + 3h + 3h^2 + h^3) - 2}{h} = \lim_{h \to 0} \frac{3h - h^3}{h} = \lim_{h \to 0} \frac{h(3 - h^2)}{h}$$

$$= \lim_{h \to 0} (3 - h^2) = 3 - 0 = 3$$

Tangent line: $y - 2 = 3(x - 1) \quad \Leftrightarrow \quad y - 2 = 3x - 3 \quad \Leftrightarrow \quad y = 3x - 1$

29. Use (4) with $f(t) = (2t+1)/(t+3)$.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\dfrac{2(a+h)+1}{(a+h)+3} - \dfrac{2a+1}{a+3}}{h} = \lim_{h \to 0} \frac{(2a+2h+1)(a+3) - (2a+1)(a+h+3)}{h(a+h+3)(a+3)}$$

$$= \lim_{h \to 0} \frac{(2a^2 + 6a + 2ah + 6h + a + 3) - (2a^2 + 2ah + 6a + a + h + 3)}{h(a+h+3)(a+3)}$$

$$= \lim_{h \to 0} \frac{5h}{h(a+h+3)(a+3)} = \lim_{h \to 0} \frac{5}{(a+h+3)(a+3)} = \frac{5}{(a+3)^2}$$

32) se (4) with $f(x) = \dfrac{4}{\sqrt{1-x}}$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\dfrac{4}{\sqrt{1-(a+h)}} - \dfrac{4}{\sqrt{1-a}}}{h}$$

$$= 4 \lim_{h\to 0} \frac{\dfrac{\sqrt{1-a}-\sqrt{1-a-h}}{\sqrt{1-a-h}\,\sqrt{1-a}}}{h} = 4 \lim_{h\to 0} \frac{\sqrt{1-a}-\sqrt{1-a-h}}{h\sqrt{1-a-h}\,\sqrt{1-a}}$$

$$= 4 \lim_{h\to 0} \frac{\sqrt{1-a}-\sqrt{1-a-h}}{h\sqrt{1-a-h}\,\sqrt{1-a}} \cdot \frac{\sqrt{1-a}+\sqrt{1-a-h}}{\sqrt{1-a}+\sqrt{1-a-h}} = 4 \lim_{h\to 0} \frac{(\sqrt{1-a})^2-(\sqrt{1-a-h})^2}{h\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a}+\sqrt{1-a-h})}$$

$$= 4 \lim_{h\to 0} \frac{(1-a)-(1-a-h)}{h\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a}+\sqrt{1-a-h})} = 4 \lim_{h\to 0} \frac{-h}{h\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a}+\sqrt{1-a-h})}$$

$$= 4 \lim_{h\to 0} \frac{-1}{\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a}+\sqrt{1-a-h})} = 4 \cdot \frac{-1}{\sqrt{1-a}\,\sqrt{1-a}(\sqrt{1-a}+\sqrt{1-a})}$$

$$= \frac{-4}{(1-a)(2\sqrt{1-a})} = \frac{-2}{(1-a)^1(1-a)^{1/2}} = \frac{-2}{(1-a)^{3/2}}$$

35. By Equation 5, $\displaystyle\lim_{x\to 5} \frac{2^x - 32}{x-5} = f'(5)$, where $f(x) = 2^x$ and $a = 5$.

37. By (4), $\displaystyle\lim_{h\to 0} \frac{\cos(\pi+h)+1}{h} = f'(\pi)$, where $f(x) = \cos x$ and $a = \pi$.

Or: By (4), $\displaystyle\lim_{h\to 0} \frac{\cos(\pi+h)+1}{h} = f'(0)$, where $f(x) = \cos(\pi + x)$ and $a = 0$.