Homework 5 – Solutions

Section 2.7

3. (a)$'$ = II, since from left to right, the slopes of the tangents to graph (a) start out negative, become 0, then positive, then 0, then negative again. The actual function values in graph II follow the same pattern.

(b)$'$ = IV, since from left to right, the slopes of the tangents to graph (b) start out at a fixed positive quantity, then suddenly become negative, then positive again. The discontinuities in graph IV indicate sudden changes in the slopes of the tangents.

(c)$'$ = I, since the slopes of the tangents to graph (c) are negative for $x < 0$ and positive for $x > 0$, as are the function values of graph I.

(d)$'$ = III, since from left to right, the slopes of the tangents to graph (d) are positive, then 0, then negative, then 0, then positive, then 0, then negative again, and the function values in graph III follow the same pattern.

21. $$f'(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} = \lim_{h\to 0} \frac{[5(t+h) - 9(t+h)^2] - (5t - 9t^2)}{h}$$

$$= \lim_{h\to 0} \frac{5t + 5h - 9(t^2 + 2th + h^2) - 5t + 9t^2}{h} = \lim_{h\to 0} \frac{5t + 5h - 9t^2 - 18th - 9h^2 - 5t + 9t^2}{h}$$

$$= \lim_{h\to 0} \frac{5h - 18th - 9h^2}{h} = \lim_{h\to 0} \frac{h(5 - 18t - 9h)}{h} = \lim_{h\to 0} (5 - 18t - 9h) = 5 - 18t$$

Domain of f = domain of $f' = \mathbb{R}$.

24. $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x + h + \sqrt{x+h}) - (x + \sqrt{x})}{h}$$

$$= \lim_{h\to 0} \left(\frac{h}{h} + \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) = \lim_{h\to 0} \left[1 + \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \right]$$

$$= \lim_{h\to 0} \left(1 + \frac{1}{\sqrt{x+h} + \sqrt{x}} \right) = 1 + \frac{1}{\sqrt{x} + \sqrt{x}} = 1 + \frac{1}{2\sqrt{x}}$$

Domain of $f = [0, \infty)$, domain of $f' = (0, \infty)$.

26. $f(x) = \frac{3+x}{1-3x}$

f'(x)=$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{x+h-\sqrt{x+h}-(x+h)}{h} = \lim_{h\to 0} \frac{\frac{3+x+h}{1-3(x+h)} - \frac{3+x}{1-3x}}{h}$

$$= \lim_{h\to 0} \frac{\frac{(3+x+h)(1-3x) - (3+x)(1-3(x+h))}{(1-3(x+h))(1-3x)}}{h} = \lim_{h\to 0} \frac{10h}{h(1-3(x+h))(1-3x)} = \frac{10}{(1-3x)^2}$$

Domain of f = Domain of f' = R-$\{1/3\}$

28. $g'(t) = \lim_{h\to 0} \frac{g(t+h)-g(t)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{t+h}} - \frac{1}{\sqrt{t}}}{h} = \lim_{h\to 0} \frac{\frac{\sqrt{t}-\sqrt{t+h}}{\sqrt{t+h}\,\sqrt{t}}}{h} = \lim_{h\to 0} \left(\frac{\sqrt{t}-\sqrt{t+h}}{h\,\sqrt{t+h}\,\sqrt{t}} \cdot \frac{\sqrt{t}+\sqrt{t+h}}{\sqrt{t}+\sqrt{t+h}} \right)$

$$= \lim_{h\to 0} \frac{t-(t+h)}{h\sqrt{t+h}\sqrt{t}\,(\sqrt{t}+\sqrt{t+h})} = \lim_{h\to 0} \frac{-h}{h\sqrt{t+h}\sqrt{t}\,(\sqrt{t}+\sqrt{t+h})} = \lim_{h\to 0} \frac{-1}{\sqrt{t+h}\sqrt{t}\,(\sqrt{t}+\sqrt{t+h})}$$

$$= \frac{-1}{\sqrt{t}\sqrt{t}\,(\sqrt{t}+\sqrt{t})} = \frac{-1}{t\,(2\sqrt{t})} = -\frac{1}{2t^{3/2}}$$

Domain of g = domain of $g' = (0, \infty)$.

38.

38 f is not differentiable at $x = -1$, because there is a discontinuity there, and at $x = 2$, because the graph has a corner there.

41. $a = f, b = f', c = f''$. We can see this because where a has a horizontal tangent, $b = 0$, and where b has a horizontal tangent, $c = 0$. We can immediately see that c can be neither f nor f', since at the points where c has a horizontal tangent, neither a nor b is equal to 0.

46. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^3 - 3(x+h)] - (x^3-3x)}{h}$

$$= \lim_{h\to 0} \frac{(x^3+3x^2h+3xh^2+h^3-3x-3h)-(x^3-3x)}{h} = \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3-3h}{h}$$

$$= \lim_{h\to 0} \frac{h(3x^2+3xh+h^2-3)}{h} = \lim_{h\to 0}(3x^2+3xh+h^2-3) = 3x^2-3$$

$f''=\lim_{h\to 0} \frac{3(x+h)^2-3(x+h)-3x^2+3}{h} = \lim_{h\to 0} \frac{3x^2+6hx+3h^2-3x^2}{h} = 6x$

50. (a) $g'(0) = \lim_{x\to 0} \frac{g(x)-g(0)}{x-0} = \lim_{x\to 0} \frac{x^{2/3}-0}{x} = \lim_{x\to 0} \frac{1}{x^{1/3}}$, which does not exist.

(b) $g'(a) = \lim_{x\to a} \frac{g(x)-g(a)}{x-a} = \lim_{x\to a} \frac{x^{2/3}-a^{2/3}}{x-a} = \lim_{x\to a} \frac{(x^{1/3}-a^{1/3})(x^{1/3}+a^{1/3})}{(x^{1/3}-a^{1/3})(x^{2/3}+x^{1/3}a^{1/3}+a^{2/3})}$

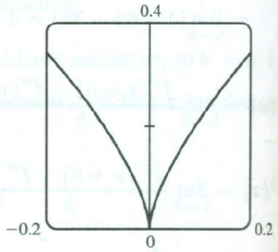
$$= \lim_{x\to a} \frac{x^{1/3}+a^{1/3}}{x^{2/3}+x^{1/3}a^{1/3}+a^{2/3}} = \frac{2a^{1/3}}{3a^{2/3}} = \frac{2}{3a^{1/3}} \text{ or } \tfrac{2}{3}a^{-1/3}$$

(c) $g(x) = x^{2/3}$ is continuous at $x = 0$ and

$\lim_{x\to 0} |g'(x)| = \lim_{x\to 0} \frac{2}{3\,|x|^{1/3}} = \infty$. This shows that

g has a vertical tangent line at $x = 0$.

(d)



Section 2.8

5. The derivative f' is increasing when the slopes of the tangent lines of f are becoming larger as x increases. This seems to be the case on the interval $(2,5)$. The derivative is decreasing when the slopes of the tangent lines of f are becoming smaller as x increases, and this seems to be the case on $(-\infty, 2)$ and $(5, \infty)$. So f' is increasing on $(2,5)$ and decreasing on $(-\infty, 2)$ and $(5, \infty)$.

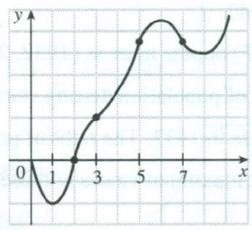
16. (a) f is increasing where f' is positive, on $(1,6)$ and $(8,\infty)$, and decreasing where f' is negative, on $(0,1)$ and $(6,8)$.

(b) f has a local maximum where f' changes from positive to negative, at $x=6$, and local minima where f' changes from negative to positive, at $x=1$ and at $x=8$.

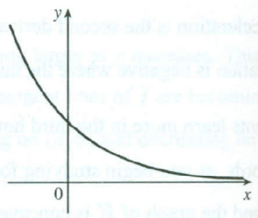
(c) f is concave upward where f' is increasing, that is, on $(0,2)$, $(3,5)$, and $(7,\infty)$, and concave downward where f' is decreasing, that is, on $(2,3)$ and $(5,7)$.

(d) There are points of inflection where f changes its direction of concavity, at $x=2$, $x=3$, $x=5$ and $x=7$.

(e)



18. The function must be always decreasing and concave upward.



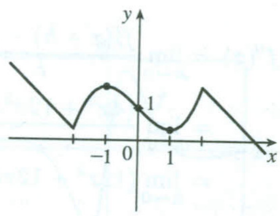
22. $f'(1) = f'(-1) = 0 \Rightarrow$ horizontal tangents at $x = \pm 1$.

$f'(x) < 0$ if $|x| < 1 \Rightarrow f$ is decreasing on $(-1, 1)$.

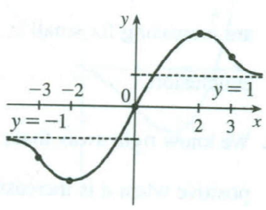
$f'(x) > 0$ if $1 < |x| < 2 \Rightarrow f$ is increasing on $(-2,-1)$ and $(1,2)$.

$f'(x) = -1$ if $|x| > 2 \Rightarrow$ the graph of f has constant slope -1 on $(-\infty, -2)$ and $(2, \infty)$.

$f''(x) < 0$ if $-2 < x < 0 \Rightarrow f$ is concave downward on $(-2, 0)$. The point $(0,1)$ is an inflection point.



24. $f'(x) > 0$ if $|x| < 2 \Rightarrow f$ is increasing on $(-2,2)$. $f'(x) < 0$ if $|x| > 2 \Rightarrow$ f is decreasing on $(-\infty, -2)$ and $(2, \infty)$. $f'(2) = 0$, so f has a horizontal tangent (and local maximum) at $x = 2$. $\lim_{x\to\infty} f(x) = 1 \Rightarrow y = 1$ is a horizontal asymptote. $f(-x) = -f(x) \Rightarrow f$ is an odd function (its graph is symmetric about the origin). Finally, $f''(x) < 0$ if $0 < x < 3$ and $f''(x) > 0$ if $x > 3$, so f is CD on $(0,3)$ and CU on $(3, \infty)$.



25. (a) Since e^{-x^2} is positive for all x, $f'(x) = xe^{-x^2}$ is positive where $x > 0$ and negative where $x < 0$. Thus, f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.

(b) Since f changes from decreasing to increasing at $x = 0$, f has a minimum value there.

28. (a) $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{[(x+h)^4 - 2(x+h)^2] - (x^4 - 2x^2)}{h}$

$$= \lim_{h \to 0} \frac{(x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - 2x^2 - 4xh - 2h^2) - (x^4 - 2x^2)}{h}$$

$$= \lim_{h \to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4 - 4xh - 2h^2}{h} = \lim_{h \to 0} (4x^3 + 6x^2h + 4xh^2 + h^3 - 4x - 2h) = 4x^3 - 4x$$

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{[4(x+h)^3 - 4(x+h)] - (4x^3 - 4x)}{h}$$

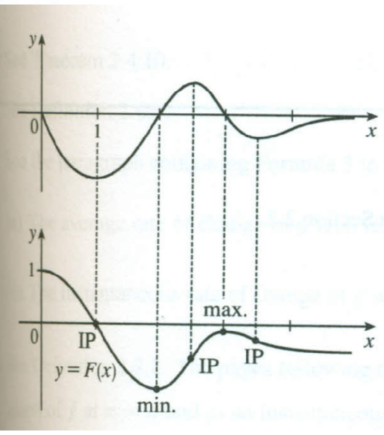
$$= \lim_{h \to 0} \frac{(4x^3 + 12x^2h + 12xh^2 + 4h^3 - 4x - 4h) - (4x^3 - 4x)}{h} = \lim_{h \to 0} \frac{12x^2h + 12xh^2 + 4h^3 - 4h}{h}$$

$$= \lim_{h \to 0} (12x^2 + 12xh + 4h^2 - 4) = 12x^2 - 4$$

(b) $f'(x) > 0 \iff 4x^3 - 4x > 0 \iff 4x(x^2 - 1) > 0 \iff 4x(x+1)(x-1) > 0$, so f is increasing on $(-1, 0)$ and $(1, \infty)$ and f is decreasing on $(-\infty, -1)$ and $(0, 1)$.

(c) $f''(x) > 0 \iff 12x^2 - 4 > 0 \iff 12x^2 > 4 \iff x^2 > \frac{1}{3} \iff |x| > \sqrt{\frac{1}{3}}$, so f is CU on $\left(-\infty, -\sqrt{\frac{1}{3}}\right)$ and $\left(\sqrt{\frac{1}{3}}, \infty\right)$ and f is CD on $\left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right)$.

31.



The graph of F must start at $(0, 1)$. Where the given graph, $y = f(x)$, has a local minimum or maximum, the graph of F will have an inflection point.

Where f is negative (positive), F is decreasing (increasing).

Where f changes from negative to positive, F will have a minimum.

Where f changes from positive to negative, F will have a maximum.

Where f is decreasing (increasing), F is concave downward (upward).

Section 3.1

10.

$h(x) = (x-2)(2x+3) = 2x^2 - x - 6 \quad \Rightarrow \quad h'(x) = 2(2x) - 1 - 0 = 4x - 1$

14.

$h(t) = \sqrt[4]{t} - 4e^t = t^{1/4} - 4e^t \quad \Rightarrow \quad h'(t) = \frac{1}{4}t^{-3/4} - 4(e^t) = \frac{1}{4}t^{-3/4} - 4e^t$

18. $f(x) = \dfrac{x^2 - 3x + 1}{x^2} = 1 - \dfrac{3}{x} + \dfrac{1}{x^2} = 1 - 3x^{-1} + x^{-2} \quad\Rightarrow$

$f'(x) = 0 - 3(-1)x^{-2} + (-2)x^{-3} = 3x^{-2} - 2x^{-3}$ or $\dfrac{3}{x^2} - \dfrac{2}{x^3}$ or $\dfrac{3x-2}{x^3}$

19. $y = \dfrac{x^2 + 4x + 3}{\sqrt{x}} = x^{3/2} + 4x^{1/2} + 3x^{-1/2} \quad\Rightarrow$

$y' = \frac{3}{2}x^{1/2} + 4\left(\frac{1}{2}\right)x^{-1/2} + 3\left(-\frac{1}{2}\right)x^{-3/2} = \frac{3}{2}\sqrt{x} + \dfrac{2}{\sqrt{x}} - \dfrac{3}{2x\sqrt{x}}$ $\left[\text{note that } x^{3/2} = x^{2/2} \cdot x^{1/2} = x\sqrt{x}\right]$

The last expression can be written as $\dfrac{3x^2}{2x\sqrt{x}} + \dfrac{4x}{2x\sqrt{x}} - \dfrac{3}{2x\sqrt{x}} = \dfrac{3x^2 + 4x - 3}{2x\sqrt{x}}$.

24. $v = \left(\sqrt{x} + \dfrac{1}{\sqrt[3]{x}}\right)^2 = \left(\sqrt{x}\right)^2 + 2\sqrt{x}\cdot\dfrac{1}{\sqrt[3]{x}} + \left(\dfrac{1}{\sqrt[3]{x}}\right)^2 = x + 2x^{1/2-1/3} + 1/x^{2/3} = x + 2x^{1/6} + x^{-2/3} \quad\Rightarrow$

$v' = 1 + 2\left(\frac{1}{6}x^{-5/6}\right) - \frac{2}{3}x^{-5/3} = 1 + \frac{1}{3}x^{-5/6} - \frac{2}{3}x^{-5/3}$ or $1 + \dfrac{1}{3\sqrt[6]{x^5}} - \dfrac{2}{3\sqrt[3]{x^5}}$

28. $y = x^4 + 2x^2 - x \quad\Rightarrow\quad y' = 4x^3 + 4x - 1$. At $(1, 2)$, $y' = 7$ and an equation of the tangent line is $y - 2 = 7(x - 1)$ or $y = 7x - 5$.

30. $y = (1 + 2x)^2 = 1 + 4x + 4x^2 \quad\Rightarrow\quad y' = 4 + 8x$. At $(1, 9)$, $y' = 12$ and an equation of the tangent line is $y - 9 = 12(x - 1)$ or $y = 12x - 3$. The slope of the normal line is $-\frac{1}{12}$ (the negative reciprocal of 12) and an equation of the normal line is $y - 9 = -\frac{1}{12}(x - 1)$ or $y = -\frac{1}{12}x + \frac{109}{12}$.

42. $G(r) = \sqrt{r} + \sqrt[3]{r} \quad\Rightarrow\quad G'(r) = \frac{1}{2}r^{-1/2} + \frac{1}{3}r^{-2/3} \quad\Rightarrow\quad G''(r) = -\frac{1}{4}r^{-3/2} - \frac{2}{9}r^{-5/3}$

48. $f(x) = x^3 - 4x^2 + 5x \quad\Rightarrow\quad f'(x) = 3x^2 - 8x + 5 \quad\Rightarrow\quad f''(x) = 6x - 8$.

$f''(x) > 0 \quad\Rightarrow\quad 6x - 8 > 0 \quad\Rightarrow\quad x > \frac{4}{3}$. f is concave upward when $f''(x) > 0$; that is, on $\left(\frac{4}{3}, \infty\right)$.

50. $f(x) = x^3 + 3x^2 + x + 3$ has a horizontal tangent when $f'(x) = 3x^2 + 6x + 1 = 0 \quad\Leftrightarrow$

$x = \dfrac{-6 \pm \sqrt{36 - 12}}{6} = -1 \pm \frac{1}{3}\sqrt{6}$.

51. $y = 6x^3 + 5x - 3 \quad\Rightarrow\quad m = y' = 18x^2 + 5$, but $x^2 \geq 0$ for all x, so $m \geq 5$ for all x.

53. The slope of the line $12x - y = 1$ (or $y = 12x - 1$) is 12, so the slope of both lines tangent to the curve is 12. $y = 1 + x^3 \quad\Rightarrow\quad y' = 3x^2$. Thus, $3x^2 = 12 \quad\Rightarrow\quad x^2 = 4 \quad\Rightarrow\quad x = \pm 2$, which are the x-coordinates at which the tangent lines have slope 12. The points on the curve are $(2, 9)$ and $(-2, -7)$, so the tangent line equations are $y - 9 = 12(x - 2)$ or $y = 12x - 15$ and $y + 7 = 12(x + 2)$ or $y = 12x + 17$.

66. $y = x^4 + ax^3 + bx^2 + cx + d \quad\Rightarrow\quad y(0) = d$. Since the tangent line $y = 2x + 1$ is equal to 1 at $x = 0$, we must have $d = 1$. $y' = 4x^3 + 3ax^2 + 2bx + c \quad\Rightarrow\quad y'(0) = c$. Since the slope of the tangent line $y = 2x + 1$ at $x = 0$ is 2, we must have $c = 2$. Now $y(1) = 1 + a + b + c + d = a + b + 4$ and the tangent line $y = 2 - 3x$ at $x = 1$ has y-coordinate -1, so $a + b + 4 = -1$ or $a + b = -5$ **(1)**. Also, $y'(1) = 4 + 3a + 2b + c = 3a + 2b + 6$ and the slope of the tangent line $y = 2 - 3x$ at $x = 1$ is -3, so $3a + 2b + 6 = -3$ or $3a + 2b = -9$ **(2)**. Adding -2 times **(1)** to **(2)** gives us $a = 1$ and hence, $b = -6$. The curve has equation $y = x^4 + x^3 - 6x^2 + 2x + 1$.

68.

68. The slope of the curve $y = c\sqrt{x}$ is $y' = \dfrac{c}{2\sqrt{x}}$ and the slope of the tangent line $y = \frac{3}{2}x + 6$ is $\frac{3}{2}$. These must be equal at the point of tangency $\left(a, c\sqrt{a}\right)$, so $\dfrac{c}{2\sqrt{a}} = \dfrac{3}{2} \Rightarrow c = 3\sqrt{a}$. The y-coordinates must be equal at $x = a$, so $c\sqrt{a} = \frac{3}{2}a + 6 \Rightarrow \left(3\sqrt{a}\right)\sqrt{a} = \frac{3}{2}a + 6 \Rightarrow 3a = \frac{3}{2}a + 6 \Rightarrow \frac{3}{2}a = 6 \Rightarrow a = 4$. Since $c = 3\sqrt{a}$, we have $c = 3\sqrt{4} = 6$.

73. $y = x^2 \Rightarrow y' = 2x$, so the slope of a tangent line at the point (a, a^2) is $y' = 2a$ and the slope of a normal line is $-1/(2a)$, for $a \neq 0$. The slope of the normal line through the points (a, a^2) and $(0, c)$ is $\dfrac{a^2 - c}{a - 0}$, so $\dfrac{a^2 - c}{a} = -\dfrac{1}{2a} \Rightarrow a^2 - c = -\frac{1}{2} \Rightarrow a^2 = c - \frac{1}{2}$. The last equation has two solutions if $c > \frac{1}{2}$, one solution if $c = \frac{1}{2}$, and no solution if $c < \frac{1}{2}$. Since the y-axis is normal to $y = x^2$ regardless of the value of c (this is the case for $a = 0$), we have three normal lines if $c > \frac{1}{2}$ and one normal line if $c \leq \frac{1}{2}$.