Homework 6 – Solutions

Section 3.2

6.

By the Quotient Rule, $y = \dfrac{e^x}{1+x} \quad \Rightarrow \quad y' = \dfrac{(1+x)\,e^x - e^x(1)}{(1+x)^2} = \dfrac{e^x + xe^x - e^x}{(x+1)^2} = \dfrac{xe^x}{(x+1)^2}$.

9.

$F(y) = \left(\dfrac{1}{y^2} - \dfrac{3}{y^4}\right)(y + 5y^3) = (y^{-2} - 3y^{-4})(y + 5y^3) \quad \overset{\text{PR}}{\Rightarrow}$

$$F'(y) = (y^{-2} - 3y^{-4})(1 + 15y^2) + (y + 5y^3)(-2y^{-3} + 12y^{-5})$$
$$= (y^{-2} + 15 - 3y^{-4} - 45y^{-2}) + (-2y^{-2} + 12y^{-4} - 10 + 60y^{-2})$$
$$= 5 + 14y^{-2} + 9y^{-4} \text{ or } 5 + 14/y^2 + 9/y^4$$

10. $R(t) = (t + e^t)(3 - \sqrt{t}) \quad \overset{\text{PR}}{\Rightarrow}$

$$R'(t) = (t + e^t)\left(-\tfrac{1}{2}t^{-1/2}\right) + (3 - \sqrt{t})(1 + e^t)$$
$$= \left(-\tfrac{1}{2}t^{1/2} - \tfrac{1}{2}t^{-1/2}e^t\right) + (3 + 3e^t - \sqrt{t} - \sqrt{t}\,e^t) = 3 + 3e^t - \tfrac{3}{2}\sqrt{t} - \sqrt{t}\,e^t - e^t/(2\sqrt{t})$$

20. $g(t) = \dfrac{t - \sqrt{t}}{t^{1/3}} = \dfrac{t}{t^{1/3}} - \dfrac{t^{1/2}}{t^{1/3}} = t^{2/3} - t^{1/6} \quad \Rightarrow \quad g'(t) = \tfrac{2}{3}t^{-1/3} - \tfrac{1}{6}t^{-5/6}$

22. $f(x) = \dfrac{1 - xe^x}{x + e^x} \quad \overset{\text{QR}}{\Rightarrow} \quad f'(x) = \dfrac{(x + e^x)(-xe^x)' - (1 - xe^x)(1 + e^x)}{(x + e^x)^2}$

$$\overset{\text{PR}}{\Rightarrow} \quad f'(x) = \frac{(x + e^x)[-(xe^x + e^x \cdot 1)] - (1 + e^x - xe^x - xe^{2x})}{(x + e^x)^2}$$
$$= \frac{-x^2e^x - xe^x - xe^{2x} - e^{2x} - 1 - e^x + xe^x + xe^{2x}}{(x + e^x)^2} = \frac{-x^2e^x - e^{2x} - e^x - 1}{(x + e^x)^2}$$

31. $y = 2xe^x \quad \Rightarrow \quad y' = 2(x \cdot e^x + e^x \cdot 1) = 2e^x(x + 1)$.

At $(0, 0)$, $y' = 2e^0(0 + 1) = 2 \cdot 1 \cdot 1 = 2$, and an equation of the tangent line is $y - 0 = 2(x - 0)$, or $y = 2x$. The slope of the normal line is $-\tfrac{1}{2}$, so an equation of the normal line is $y - 0 = -\tfrac{1}{2}(x - 0)$, or $y = -\tfrac{1}{2}x$.

40.

$g(x) = \frac{x}{e^x} \quad \Rightarrow \quad g'(x) = \frac{e^x \cdot 1 - x \cdot e^x}{(e^x)^2} = \frac{e^x(1-x)}{(e^x)^2} = \frac{1-x}{e^x} \quad \Rightarrow$

$g''(x) = \frac{e^x \cdot (-1) - (1-x)e^x}{(e^x)^2} = \frac{e^x[-1-(1-x)]}{(e^x)^2} = \frac{x-2}{e^x} \quad \Rightarrow$

$g'''(x) = \frac{e^x \cdot 1 - (x-2)e^x}{(e^x)^2} = \frac{e^x[1-(x-2)]}{(e^x)^2} = \frac{3-x}{e^x} \quad \Rightarrow$

$g^{(4)}(x) = \frac{e^x \cdot (-1) - (3-x)e^x}{(e^x)^2} = \frac{e^x[-1-(3-x)]}{(e^x)^2} = \frac{x-4}{e^x}.$

The pattern suggests that $g^{(n)}(x) = \frac{(x-n)(-1)^n}{e^x}$. (We could use mathematical induction to prove this formula.)

42. We are given that $f(2) = -3$, $g(2) = 4$, $f'(2) = -2$, and $g'(2) = 7$.

(a) $h(x) = 5f(x) - 4g(x) \quad \Rightarrow \quad h'(x) = 5f'(x) - 4g'(x)$, so

$h'(2) = 5f'(2) - 4g'(2) = 5(-2) - 4(7) = -10 - 28 = -38.$

(b) $h(x) = f(x)g(x) \quad \Rightarrow \quad h'(x) = f(x)g'(x) + g(x)f'(x)$, so

$h'(2) = f(2)g'(2) + g(2)f'(2) = (-3)(7) + (4)(-2) = -21 - 8 = -29.$

(c) $h(x) = \frac{f(x)}{g(x)} \quad \Rightarrow \quad h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$, so

$h'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{[g(2)]^2} = \frac{4(-2) - (-3)(7)}{4^2} = \frac{-8+21}{16} = \frac{13}{16}.$

(d) $h(x) = \frac{g(x)}{1+f(x)} \quad \Rightarrow \quad h'(x) = \frac{[1+f(x)]\,g'(x) - g(x)f'(x)}{[1+f(x)]^2}$, so

$h'(2) = \frac{[1+f(2)]\,g'(2) - g(2)\,f'(2)}{[1+f(x)]^2} = \frac{[1+(-3)](7) - 4(-2)}{[1+(-3)]^2} = \frac{-14+8}{(-2)^2} = \frac{-6}{4} = -\frac{3}{2}.$

44. $\frac{d}{dx}\left[\frac{h(x)}{x}\right] = \frac{xh'(x) - h(x)\cdot 1}{x^2} \quad \Rightarrow \quad \frac{d}{dx}\left[\frac{h(x)}{x}\right]_{x=2} = \frac{2h'(2) - h(2)}{2^2} = \frac{2(-3) - (4)}{4} = \frac{-10}{4} = -2.5$

53. If $y = f(x) = \frac{x}{x+1}$, then $f'(x) = \frac{(x+1)(1) - x(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$. When $x = a$, the equation of the tangent line is

$y - \frac{a}{a+1} = \frac{1}{(a+1)^2}(x-a)$. This line passes through $(1,2)$ when $2 - \frac{a}{a+1} = \frac{1}{(a+1)^2}(1-a) \quad \Leftrightarrow$

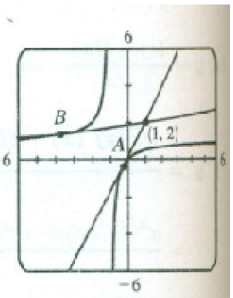
$2(a+1)^2 - a(a+1) = 1 - a \quad \Leftrightarrow \quad 2a^2 + 4a + 2 - a^2 - a - 1 + a = 0 \quad \Leftrightarrow \quad a^2 + 4a + 1 = 0.$

The quadratic formula gives the roots of this equation as $a = \frac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2(1)} = \frac{-4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3}$.

so there are two such tangent lines. Since

$$f(-2\pm\sqrt{3})=\frac{-2\pm\sqrt{3}}{-2\pm\sqrt{3}+1}=\frac{-2\pm\sqrt{3}}{-1\pm\sqrt{3}}\cdot\frac{-1\mp\sqrt{3}}{-1\mp\sqrt{3}}$$

$$=\frac{2\pm2\sqrt{3}\mp\sqrt{3}-3}{1-3}=\frac{-1\pm\sqrt{3}}{-2}=\frac{1\mp\sqrt{3}}{2},$$



the lines touch the curve at $A\left(-2+\sqrt{3},\frac{1-\sqrt{3}}{2}\right)\approx(-0.27,-0.37)$

and $B\left(-2-\sqrt{3},\frac{1+\sqrt{3}}{2}\right)\approx(-3.73,1.37)$.

Section 3.3

4. $\mathbf{f(x)=x.\,sinx}$

$\mathbf{f'(x)=sinx+x.\,cosx=sinx+xcosx}$

6. $g(\theta)=e^{\theta}(\tan\theta-\theta)\quad\Rightarrow\quad g'(\theta)=e^{\theta}(\sec^2\theta-1)+(\tan\theta-\theta)e^{\theta}=e^{\theta}(\sec^2\theta-1+\tan\theta-\theta)$

12. $y=\dfrac{1-\sec x}{\tan x}\quad\Rightarrow$

$$y'=\frac{\tan x(-\sec x\tan x)-(1-\sec x)(\sec^2 x)}{(\tan x)^2}=\frac{\sec x\left(-\tan^2 x-\sec x+\sec^2 x\right)}{\tan^2 x}=\frac{\sec x(1-\sec x)}{\tan^2 x}$$

22.

22. $y=\dfrac{1}{\sin x+\cos x}\quad\Rightarrow\quad y'=-\dfrac{\cos x-\sin x}{(\sin x+\cos x)^2}$ [Reciprocal Rule]. At $(0,1)$, $y'=-\dfrac{1-0}{(0+1)^2}=-1$, and an equation of the tangent line is $y-1=-1(x-0)$, or $y=-x+1$.

39. $\dfrac{d}{dx}(\sin x)=\cos x\;\Rightarrow\;\dfrac{d^2}{dx^2}(\sin x)=-\sin x\;\Rightarrow\;\dfrac{d^3}{dx^3}(\sin x)=-\cos x\;\Rightarrow\;\dfrac{d^4}{dx^4}(\sin x)=\sin x.$

The derivatives of $\sin x$ occur in a cycle of four. Since $99=4(24)+3$, we have $\dfrac{d^{99}}{dx^{99}}(\sin x)=\dfrac{d^3}{dx^3}(\sin x)=-\cos x.$

45.

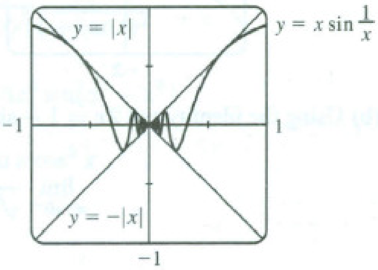
SECTION 3.3 DERIVATIVES OF

$$\lim_{\theta\to0}\frac{\sin\theta}{\theta+\tan\theta}=\frac{\displaystyle\lim_{\theta\to0}\frac{\sin\theta}{\theta}}{\displaystyle\lim_{\theta\to0}\frac{\theta+\tan\theta}{\theta}}=\frac{1}{\displaystyle\lim_{\theta\to0}\left(1+\frac{\sin\theta}{\theta}\cdot\frac{1}{\cos\theta}\right)}=\frac{1}{1+1\cdot1}=\frac{1}{2}$$

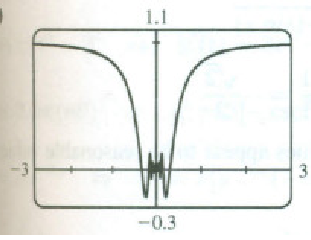
46.

(a) Let $\theta = \frac{1}{x}$. Then as $x \to \infty$, $\theta \to 0$, and $\lim_{x \to \infty} x \sin \frac{1}{x} = \lim_{\theta \to 0} \frac{1}{\theta} \sin \theta = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$.

(b) Since $-1 \leq \sin(1/x) \leq 1$, we have (as illustrated in the figure) $-|x| \leq x \sin(1/x) \leq |x|$. We know that $\lim_{x \to 0} (|x|) = 0$ and $\lim_{x \to 0} (-|x|) = 0$; so by the Squeeze Theorem, $\lim_{x \to 0} x \sin(1/x) = 0$.



(c)



Section 3.4

5. Let $u = g(x) = \sqrt{x}$ and $y = f(u) = e^u$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (e^u)\left(\frac{1}{2}x^{-1/2}\right) = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$.

6. Let $u = g(x) = 2 - e^x$ and $y = f(u) = \sqrt{u}$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\frac{1}{2}u^{-1/2})(-e^x) = -\frac{e^x}{2\sqrt{2 - e^x}}$.

12. $f(t) = \sqrt[3]{1 + \tan t} = (1 + \tan t)^{1/3} \quad \Rightarrow \quad f'(t) = \frac{1}{3}(1 + \tan t)^{-2/3} \sec^2 t = \frac{\sec^2 t}{3\sqrt[3]{(1 + \tan t)^2}}$

18.

$y = e^{-2t} \cos 4t \quad \Rightarrow \quad y' = e^{-2t}(-\sin 4t \cdot 4) + \cos 4t[e^{-2t}(-2)] = -2e^{-2t}(2 \sin 4t + \cos 4t)$

24.

$G(y) = \left(\frac{y^2}{y+1}\right)^5 \quad \Rightarrow \quad G'(y) = 5\left(\frac{y^2}{y+1}\right)^4 \cdot \frac{(y+1)(2y) - y^2(1)}{(y+1)^2} = 5 \cdot \frac{y^8}{(y+1)^4} \cdot \frac{y(2y + 2 - y)}{(y+1)^2} = \frac{5y^9(y+2)}{(y+1)^6}$

31. Using Formula 5 and the Chain Rule, $y = 2^{\sin \pi x} \quad \Rightarrow$

$$y' = 2^{\sin \pi x}(\ln 2) \cdot \frac{d}{dx}(\sin \pi x) = 2^{\sin \pi x}(\ln 2) \cdot \cos \pi x \cdot \pi = 2^{\sin \pi x}(\pi \ln 2) \cos \pi x$$

36. $y = 2^{3^{x^2}} \Rightarrow y' = 2^{3^{x^2}}(\ln 2)\frac{d}{dx}\left(3^{x^2}\right) = 2^{3^{x^2}}(\ln 2)3^{x^2}(\ln 3)(2x)$

38.

$y = \cos^2 x = (\cos x)^2 \Rightarrow y' = 2\cos x(-\sin x) = -2\cos x \sin x \Rightarrow$

$y'' = (-2\cos x)\cos x + \sin x(2\sin x) = -2\cos^2 x + 2\sin^2 x$

Note: Many other forms of the answers exist. For example, $y' = -\sin 2x$ and $y'' = -2\cos 2x$.

43.

$y = \sin(\sin x) \Rightarrow y' = \cos(\sin x)\cdot\cos x$. At $(\pi, 0)$, $y' = \cos(\sin\pi)\cdot\cos\pi = \cos(0)\cdot(-1) = 1(-1) = -1$, and an equation of the tangent line is $y - 0 = -1(x - \pi)$, or $y = -x + \pi$.

52. $h(x) = \sqrt{4 + 3f(x)} \Rightarrow h'(x) = \frac{1}{2}(4 + 3f(x))^{-1/2}\cdot 3f'(x)$, so

$h'(1) = \frac{1}{2}(4 + 3f(1))^{-1/2}\cdot 3f'(1) = \frac{1}{2}(4 + 3\cdot 7)^{-1/2}\cdot 3\cdot 4 = \frac{6}{\sqrt{25}} = \frac{6}{5}$

61. $r(x) = f(g(h(x))) \Rightarrow r'(x) = f'(g(h(x)))\cdot g'(h(x))\cdot h'(x)$, so

$r'(1) = f'(g(h(1)))\cdot g'(h(1))\cdot h'(1) = f'(g(2))\cdot g'(2)\cdot 4 = f'(3)\cdot 5\cdot 4 = 6\cdot 5\cdot 4 = 120$

62. $f(x) = xg(x^2) \Rightarrow f'(x) = xg'(x^2)\,2x + g(x^2)\cdot 1 = 2x^2 g'(x^2) + g(x^2) \Rightarrow$

$f''(x) = 2x^2 g''(x^2)\,2x + g'(x^2)\,4x + g'(x^2)\,2x = 4x^3 g''(x^2) + 4xg'(x^2) + 2xg'(x^2) = 6xg'(x^2) + 4x^3 g''(x^2)$