Homework 7 – Solutions

Section 3.5

8. $\frac{d}{dx}(y^5 + x^2y^3)=$

$$\frac{d}{dx}\left(1 + ye^{x^2}\right) \quad \Rightarrow \quad 5y^4\,y' + (x^2 \cdot 3y^2\,y' + y^3 \cdot 2x) = 0 + y \cdot e^{x^2} \cdot 2x + e^{x^2} \cdot y' \quad \Rightarrow$$

$$y'\left(5y^4 + 3x^2y^2 - e^{x^2}\right) = 2xye^{x^2} - 2xy^3$$

$$\Rightarrow \quad y' = \frac{2xy\left(e^{x^2} - y^2\right)}{5y^4 + 3x^2y^2 - e^{x^2}}$$

10. $\frac{d}{dx}(1+x) = \frac{d}{dx}\left[\sin(xy^2)\right] \quad \Rightarrow \quad 1 = [\cos(xy^2)](x \cdot 2y\,y' + y^2 \cdot 1) \quad \Rightarrow \quad 1 = 2xy\cos(xy^2)\,y' + y^2\cos(xy^2)$

$$1 - y^2\cos(xy^2) = 2xy\cos(xy^2)\,y' \quad \Rightarrow \quad y' = \frac{1 - y^2\cos(xy^2)}{2xy\cos(xy^2)}$$

13. $\frac{d}{dx}(e^{x/y}) = \frac{d}{dx}(x - y) \quad \Rightarrow \quad e^{x/y} \cdot \frac{d}{dx}\left(\frac{x}{y}\right) = 1 - y' \quad \Rightarrow$

$$e^{x/y} \cdot \frac{y \cdot 1 - x \cdot y'}{y^2} = 1 - y' \quad \Rightarrow \quad e^{x/y} \cdot \frac{1}{y} - \frac{xe^{x/y}}{y^2} \cdot y' = 1 - y' \quad \Rightarrow \quad y' - \frac{xe^{x/y}}{y^2} \cdot y' = 1 - \frac{e^{x/y}}{y} \quad \Rightarrow$$

$$y'\left(1 - \frac{xe^{x/y}}{y^2}\right) = \frac{y - e^{x/y}}{y} \quad \Rightarrow \quad y' = \frac{\dfrac{y - e^{x/y}}{y}}{\dfrac{y^2 - xe^{x/y}}{y^2}} = \frac{y(y - e^{x/y})}{y^2 - xe^{x/y}}$$

14. $\tan(x - y) = \frac{y}{1 + x^2} \quad \Rightarrow \quad (1 + x^2)\tan(x - y) = y \quad \Rightarrow \quad (1 + x^2)\sec^2(x - y) \cdot (1 - y') + \tan(x - y) \cdot 2x =$

$$(1 + x^2)\sec^2(x - y) - (1 + x^2)\sec^2(x - y) \cdot y' + 2x\tan(x - y) = y' \quad \Rightarrow$$

$$(1 + x^2)\sec^2(x - y) + 2x\tan(x - y) = \left[1 + (1 + x^2)\sec^2(x - y)\right] \cdot y' \quad \Rightarrow$$

$$y' = \frac{(1 + x^2)\sec^2(x - y) + 2x\tan(x - y)}{1 + (1 + x^2)\sec^2(x - y)}$$

18.

$\frac{d}{dx}[g(x) + x\sin g(x)] = \frac{d}{dx}(x^2) \quad \Rightarrow \quad g'(x) + x\cos g(x) \cdot g'(x) + \sin g(x) \cdot 1 = 2x.$ If $x = 0$, we have

$$g'(0) + 0 + \sin g(0) = 2(0) \quad \Rightarrow \quad g'(0) + \sin 0 = 0 \quad \Rightarrow \quad g'(0) + 0 = 0 \quad \Rightarrow \quad g'(0) = 0.$$

20.

$$\frac{d}{dy}(y\sec x) = \frac{d}{dy}(x\tan y) \Rightarrow y\cdot\sec x\,\tan x\cdot x' + \sec x\cdot 1 = x\cdot\sec^2 y + \tan y\cdot x' \Rightarrow$$

$$y\sec x\,\tan x\cdot x' - \tan y\cdot x' = x\sec^2 y - \sec x \Rightarrow (y\sec x\,\tan x - \tan y)\,x' = x\sec^2 y - \sec x \Rightarrow$$

$$x' = \frac{dx}{dy} = \frac{x\sec^2 y - \sec x}{y\sec x\,\tan x - \tan y}$$

23. $x^2 + xy + y^2 = 3$

$$\Rightarrow 2x + xy' + y\cdot 1 + 2yy' = 0 \Rightarrow xy' + 2yy' = -2x - y \Rightarrow y'(x+2y) = -2x - y \Rightarrow$$

$y' = \frac{-2x-y}{x+2y}$,

When $x = 1$ and $y = 1$, we have $y' = \dfrac{-2-1}{1+2\cdot 1} = \dfrac{-3}{3} = -1$, so an equation of the tangent line is

$y - 1 = -1(x-1)$ **or** $y = -x + 2$

32.

$$\sqrt{x} + \sqrt{y} = 1 \Rightarrow \frac{1}{2\sqrt{x}} + \frac{y'}{2\sqrt{y}} = 0 \Rightarrow y' = -\frac{\sqrt{y}}{\sqrt{x}} \Rightarrow$$

$$y'' = -\frac{\sqrt{x}\left[\frac{1}{2\sqrt{y}}\right]y' - \sqrt{y}\left[\frac{1}{2\sqrt{x}}\right]}{x} = -\frac{\sqrt{x}\left(\frac{1}{\sqrt{y}}\right)\left(-\frac{\sqrt{y}}{\sqrt{x}}\right) - \sqrt{y}\left(\frac{1}{\sqrt{x}}\right)}{2x} = \frac{1 + \frac{\sqrt{y}}{\sqrt{x}}}{2x}$$

$$= \frac{\sqrt{x}+\sqrt{y}}{2x\sqrt{x}} = \frac{1}{2x\sqrt{x}}$$ since x and y must satisfy the original equation, $\sqrt{x} + \sqrt{y} = 1$.

50.

$\sqrt{x} + \sqrt{y} = \sqrt{c} \Rightarrow \dfrac{1}{2\sqrt{x}} + \dfrac{y'}{2\sqrt{y}} = 0 \Rightarrow y' = -\dfrac{\sqrt{y}}{\sqrt{x}} \Rightarrow$ an equation of the tangent line at (x_0, y_0)

$y - y_0 = -\dfrac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)$. Now $x = 0 \Rightarrow y = y_0 - \dfrac{\sqrt{y_0}}{\sqrt{x_0}}(-x_0) = y_0 + \sqrt{x_0}\sqrt{y_0}$, so the y-intercept is

$y_0 + \sqrt{x_0}\sqrt{y_0}$. And $y = 0 \Rightarrow -y_0 = -\dfrac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0) \Rightarrow x - x_0 = \dfrac{y_0\sqrt{x_0}}{\sqrt{y_0}} \Rightarrow$

$x = x_0 + \sqrt{x_0}\sqrt{y_0}$, so the x-intercept is $x_0 + \sqrt{x_0}\sqrt{y_0}$. The sum of the intercepts is

$$\left(y_0 + \sqrt{x_0}\sqrt{y_0}\right) + \left(x_0 + \sqrt{x_0}\sqrt{y_0}\right) = x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = \left(\sqrt{x_0} + \sqrt{y_0}\right)^2 = \left(\sqrt{c}\right)^2 = c.$$

Section 3.6

17.

$y = (\tan^{-1} x)^2 \quad \Rightarrow \quad y' = 2(\tan^{-1} x)^1 \cdot \frac{d}{dx}(\tan^{-1} x) = 2\tan^{-1} x \cdot \frac{1}{1+x^2} = \frac{2\tan^{-1} x}{1+x^2}$

18.

$y = \tan^{-1}(x^2) \quad \Rightarrow \quad y' = \frac{1}{1+(x^2)^2} \cdot \frac{d}{dx}(x^2) = \frac{1}{1+x^4} \cdot 2x = \frac{2x}{1+x^4}$

20.

$F(\theta) = \arcsin\sqrt{\sin\theta} = \arcsin(\sin\theta)^{1/2} \quad \Rightarrow$

$F'(\theta) = \frac{1}{\sqrt{1-\left(\sqrt{\sin\theta}\right)^2}} \cdot \frac{d}{d\theta}(\sin\theta)^{1/2} = \frac{1}{\sqrt{1-\sin\theta}} \cdot \frac{1}{2}(\sin\theta)^{-1/2} \cdot \cos\theta = \frac{\cos\theta}{2\sqrt{1-\sin\theta}\,\sqrt{\sin\theta}}$

26. $y = \cos^{-1}(\sin^{-1} t) \quad \Rightarrow \quad y' = -\frac{1}{\sqrt{1-(\sin^{-1} t)^2}} \cdot \frac{d}{dt}\sin^{-1} t = -\frac{1}{\sqrt{1-(\sin^{-1} t)^2}} \cdot \frac{1}{\sqrt{1-t^2}}$

32.

$\tan^{-1}(xy) = 1 + x^2 y \quad \Rightarrow \quad \frac{1}{1+x^2y^2}(xy' + y \cdot 1) = 0 + x^2 y' + 2xy \quad \Rightarrow$

$y'\left(\frac{x}{1+x^2y^2} - x^2\right) = 2xy - \frac{y}{1+x^2y^2} \quad \Rightarrow$

$y' = \frac{2xy - \frac{y}{1+x^2y^2}}{\frac{x}{1+x^2y^2} - x^2} = \frac{2xy(1+x^2y^2) - y}{x - x^2(1+x^2y^2)} = \frac{y(-1-2x-2x^3y^2)}{x(1-x-x^3y^2)}.$

39. Let $t = e^x$. As $x \to \infty$, $t \to \infty$. $\lim_{x\to\infty} \arctan(e^x) = \lim_{t\to\infty} \arctan t = \frac{\pi}{2}$ by (3).

40. Let $t = \ln x$. As $x \to 0^+$, $t \to -\infty$. $\lim_{x\to 0^+} \tan^{-1}(\ln x) = \lim_{t\to -\infty} \tan^{-1} t = -\frac{\pi}{2}$ by (3).

41. (a) If $y = f^{-1}(x)$, then $f(y) = x$. Differentiating implicitly with respect to x and remembering that y is a function of x,

we get $f'(y)\frac{dy}{dx} = 1$, so $\frac{dy}{dx} = \frac{1}{f'(y)} \quad \Rightarrow \quad (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$.

(b) $f(4) = 5 \quad \Rightarrow \quad f^{-1}(5) = 4$. By part (a), $(f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))} = \frac{1}{f'(4)} = 1/\left(\frac{2}{3}\right) = \frac{3}{2}$.

Section 3.7

35. $y = \dfrac{\sin^2 x \tan^4 x}{(x^2+1)^2} \Rightarrow \ln y = \ln(\sin^2 x \tan^4 x) - \ln(x^2+1)^2 \Rightarrow$

$\ln y = \ln(\sin x)^2 + \ln(\tan x)^4 - \ln(x^2+1)^2 \Rightarrow \ln y = 2\ln|\sin x| + 4\ln|\tan x| - 2\ln(x^2+1) \Rightarrow$

$\dfrac{1}{y}y' = 2\cdot\dfrac{1}{\sin x}\cdot\cos x + 4\cdot\dfrac{1}{\tan x}\cdot\sec^2 x - 2\cdot\dfrac{1}{x^2+1}\cdot 2x \Rightarrow y' = \dfrac{\sin^2 x\tan^4 x}{(x^2+1)^2}\left(2\cot x + \dfrac{4\sec^2 x}{\tan x} - \dfrac{4x}{x^2+1}\right)$

38. $y = x^{\cos x} \Rightarrow \ln y = \ln x^{\cos x} \Rightarrow \ln y = \cos x\ln x \Rightarrow \dfrac{1}{y}y' = \cos x\cdot\dfrac{1}{x} + \ln x\cdot(-\sin x) \Rightarrow$

$y' = y\left(\dfrac{\cos x}{x} - \ln x\sin x\right) \Rightarrow y' = x^{\cos x}\left(\dfrac{\cos x}{x} - \ln x\sin x\right)$

39. $y = (\cos x)^x \Rightarrow \ln y = \ln(\cos x)^x \Rightarrow \ln y = x\ln\cos x \Rightarrow \dfrac{1}{y}y' = x\cdot\dfrac{1}{\cos x}\cdot(-\sin x) + \ln\cos x\cdot 1 \Rightarrow$

$y' = y\left(\ln\cos x - \dfrac{x\sin x}{\cos x}\right) \Rightarrow y' = (\cos x)^x(\ln\cos x - x\tan x)$

41.

$y = (\tan x)^{1/x} \Rightarrow \ln y = \ln(\tan x)^{1/x} \Rightarrow \ln y = \dfrac{1}{x}\ln\tan x \Rightarrow$

$(1/y)y' = \dfrac{1}{x}\cdot\dfrac{1}{\tan x}\cdot\sec^2 x + \ln\tan x\cdot\left(-\dfrac{1}{x^2}\right) \Rightarrow y' = y\left(\dfrac{\sec^2 x}{x\tan x} - \dfrac{\ln\tan x}{x^2}\right) \Rightarrow$

$y' = (\tan x)^{1/x}\left(\dfrac{\sec^2 x}{x\tan x} - \dfrac{\ln\tan x}{x^2}\right)$ or $y' = (\tan x)^{1/x}\cdot\dfrac{1}{x}\left(\csc x\sec x - \dfrac{\ln\tan x}{x}\right)$

42.

$y = (\sin x)^{\ln x} \Rightarrow \ln y = \ln(\sin x)^{\ln x} \Rightarrow \ln y = \ln x\cdot\ln\sin x \Rightarrow \dfrac{1}{y}y' = \ln x\cdot\dfrac{1}{\sin x}\cdot\cos x + \ln\sin x\cdot\dfrac{1}{x} \Rightarrow$

$y' = y\left(\ln x\cdot\dfrac{\cos x}{\sin x} + \dfrac{\ln\sin x}{x}\right) \Rightarrow y' = (\sin x)^{\ln x}\left(\ln x\cot x + \dfrac{\ln\sin x}{x}\right)$

44.

$x^y = y^x \Rightarrow y\ln x = x\ln y \Rightarrow y\cdot\dfrac{1}{x} + (\ln x)\cdot y' = x\cdot\dfrac{1}{y}\cdot y' + \ln y \Rightarrow y'\ln x - \dfrac{x}{y}y' = \ln y - \dfrac{y}{x} \Rightarrow$

$y' = \dfrac{\ln y - y/x}{\ln x - x/y}$

47.

If $f(x) = \ln(1+x)$, then $f'(x) = \dfrac{1}{1+x}$, so $f'(0) = 1$.

Thus, $\lim\limits_{x \to 0} \dfrac{\ln(1+x)}{x} = \lim\limits_{x \to 0} \dfrac{f(x)}{x} = \lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = f'(0) = 1.$