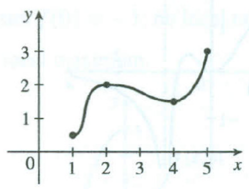
Homework 8 – Solutions

Section 4.2

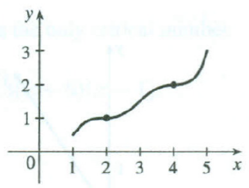
4. Absolute maximum at r; absolute minimum at a; local maxima at b and r; local minimum at d; neither a maximum nor a minimum at c and s.

6. There is no absolute maximum value; absolute minimum value is $g(4) = 1$; local maximum values are $g(3) = 4$ and $g(6) = 3$; local minimum values are $g(2) = 2$ and $g(4) = 1$.

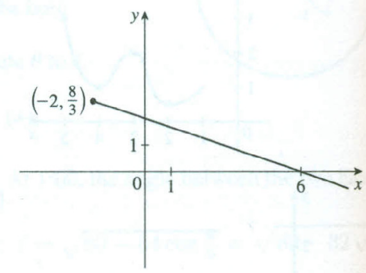
8. Absolute minimum at 1, absolute maximum at 5, local maximum at 2, local minimum at 4



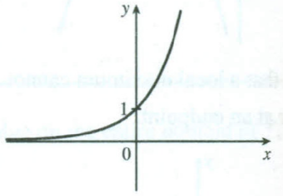
10. f has no local maximum or minimum, but 2 and 4 are critical numbers



16. $f(x) = 2 - \frac{1}{3}x$, $x \geq -2$. Absolute maximum $f(-2) = \frac{8}{3}$; no local maximum. No absolute or local minimum.



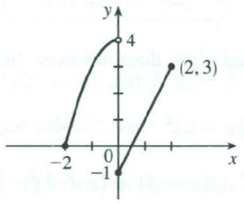
18. $f(x) = e^x$. No absolute or local maximum or minimum value.



22. $f(x) = \begin{cases} 4 - x^2 & \text{if } -2 \le x < 0 \\ 2x - 1 & \text{if } 0 \le x \le 2 \end{cases}$

Abolute minimum $f(0) = -1$; no local minimim.

No absolute or local maximum.



28. $g(t) = |3t - 4| = \begin{cases} 3t - 4 & \text{if } 3t - 4 \ge 0 \\ -(3t - 4) & \text{if } 3t - 4 < 0 \end{cases} = \begin{cases} 3t - 4 & \text{if } t \ge \frac{4}{3} \\ 4 - 3t & \text{if } t < \frac{4}{3} \end{cases}$

$g'(t) = \begin{cases} 3 & \text{if } t > \frac{4}{3} \\ -3 & \text{if } t < \frac{4}{3} \end{cases}$ and $g'(t)$ does not exist at $t = \frac{4}{3}$, so $t = \frac{4}{3}$ is a critical number.

32. $g(x) = x^{1/3} - x^{-2/3} \quad \Rightarrow \quad g'(x) = \frac{1}{3}x^{-2/3} + \frac{2}{3}x^{-5/3} = \frac{1}{3}x^{-5/3}(x + 2) = \dfrac{x + 2}{3x^{5/3}}$.

$g'(-2) = 0$ and $g'(0)$ does not exist, but 0 is not in the domain of g, so the only critical number is -2.

34. $g(\theta) = 4\theta - \tan\theta \quad \Rightarrow \quad g'(\theta) = 4 - \sec^2\theta$. $\;g'(\theta) = 0 \quad \Rightarrow \quad \sec^2\theta = 4 \quad \Rightarrow \quad \sec\theta = \pm 2 \quad \Rightarrow \quad \cos\theta = \pm\frac{1}{2} \quad \Rightarrow$

$\theta = \frac{\pi}{3} + 2n\pi$, $\frac{5\pi}{3} + 2n\pi$, $\frac{2\pi}{3} + 2n\pi$, and $\frac{4\pi}{3} + 2n\pi$ are critical numbers.

Note: The values of θ that make $g'(\theta)$ undefined are not in the domain of g.

42. $f(x) = 5 + 54x - 2x^3$, $[0, 4]$. $\;f'(x) = 54 - 6x^2 = 6(9 - x^2) = 6(3 + x)(3 - x) = 0 \quad \Leftrightarrow \quad x = -3, 3$. $\;f(0) = 5$, $f(3) = 113$, and $f(4) = 93$. So $f(3) = 113$ is the absolute maximum value and $f(0) = 5$ is the absolute minimum value.

46. $f(x) = (x^2-1)^3$, $[-1,2]$. $f'(x) = 3(x^2-1)^2(2x) = 6x(x+1)^2(x-1)^2 = 0 \Leftrightarrow x = -1, 0, 1$. $f(\pm 1) = 0$, $f(0) = -1$, and $f(2) = 27$. So $f(2) = 27$ is the absolute maximum value and $f(0) = -1$ is the absolute minimum value.

47. $f(t) = t\sqrt{4-t^2}$, $[-1,2]$.

$f'(t) = t \cdot \frac{1}{2}(4-t^2)^{-1/2}(-2t) + (4-t^2)^{1/2} \cdot 1 = \dfrac{-t^2}{\sqrt{4-t^2}} + \sqrt{4-t^2} = \dfrac{-t^2 + (4-t^2)}{\sqrt{4-t^2}} = \dfrac{4-2t^2}{\sqrt{4-t^2}}$.

$f'(t) = 0 \Rightarrow 4 - 2t^2 = 0 \Rightarrow t^2 = 2 \Rightarrow t = \pm\sqrt{2}$, but $t = -\sqrt{2}$ is not in the given interval, $[-1,2]$.

$f'(t)$ does not exist if $4 - t^2 = 0 \Rightarrow t = \pm 2$, but -2 is not in the given interval. $f(-1) = -\sqrt{3}$, $f(\sqrt{2}) = 2$, and $f(2) = 0$. So $f(\sqrt{2}) = 2$ is the absolute maximum value and $f(-1) = -\sqrt{3}$ is the absolute minimum value.

52. $f(x) = x - 2\tan^{-1}x$, $[0,4]$. $f'(x) = 1 - 2 \cdot \dfrac{1}{1+x^2} = 0 \Leftrightarrow 1 = \dfrac{2}{1+x^2} \Leftrightarrow 1 + x^2 = 2 \Leftrightarrow x^2 = 1 \Leftrightarrow$ $x = \pm 1$. $f(0) = 0$, $f(1) = 1 - \frac{\pi}{2} \approx -0.57$, and $f(4) = 4 - 2\tan^{-1}4 \approx 1.35$. So $f(4) = 4 - 2\tan^{-1}4$ is the absolute maximum value and $f(1) = 1 - \frac{\pi}{2}$ is the absolute minimum value.

53. $f(t) = 2\cos t + \sin 2t$, $[0, \pi/2]$.

$f'(t) = -2\sin t + \cos 2t \cdot 2 = -2\sin t + 2(1 - 2\sin^2 t) = -2(2\sin^2 t + \sin t - 1) = -2(2\sin t - 1)(\sin t + 1)$.

$f'(t) = 0 \Rightarrow \sin t = \frac{1}{2}$ or $\sin t = -1 \Rightarrow t = \frac{\pi}{6}$. $f(0) = 2$, $f(\frac{\pi}{6}) = \sqrt{3} + \frac{1}{2}\sqrt{3} = \frac{3}{2}\sqrt{3} \approx 2.60$, and $f(\frac{\pi}{2}) = 0$.

So $f(\frac{\pi}{6}) = \frac{3}{2}\sqrt{3}$ is the absolute maximum value and $f(\frac{\pi}{2}) = 0$ is the absolute minimum value.

Section 4.3

6. (a) f is increasing on the intervals where $f'(x) > 0$, namely, $(2,4)$ and $(6,9)$.

(b) f has a local maximum where it changes from increasing to decreasing, that is, where f' changes from positive to negative (at $x = 4$). Similarly, where f' changes from negative to positive, f has a local minimum (at $x = 2$ and at $x = 6$).

(c) When f' is increasing, its derivative f'' is positive and hence, f is concave upward. This happens on $(1,3)$, $(5,7)$, and $(8,9)$. Similarly, f is concave downward when f' is decreasing—that is, on $(0,1)$, $(3,5)$, and $(7,8)$.

(d) f has inflection points at $x = 1, 3, 5, 7$, and 8, since the direction of concavity changes at each of these values.

8. (a) $f(x) = 4x^3 + 3x^2 - 6x + 1 \Rightarrow f'(x) = 12x^2 + 6x - 6 = 6(2x^2 + x - 1) = 6(2x-1)(x+1)$. Thus, $f'(x) > 0 \Leftrightarrow x < -1$ or $x > \frac{1}{2}$ and $f'(x) < 0 \Leftrightarrow -1 < x < \frac{1}{2}$. So f is increasing on $(-\infty, -1)$ and $(\frac{1}{2}, \infty)$ and f is decreasing on $(-1, \frac{1}{2})$.

(b) f changes from increasing to decreasing at $x = -1$ and from decreasing to increasing at $x = \frac{1}{2}$. Thus, $f(-1) = 6$ is a local maximum value and $f(\frac{1}{2}) = -\frac{3}{4}$ is a local minimum value.

(c) $f''(x) = 24x + 6 = 6(4x+1)$. $f''(x) > 0 \Leftrightarrow x > -\frac{1}{4}$ and $f''(x) < 0 \Leftrightarrow x < -\frac{1}{4}$. Thus, f is concave upward on $(-\frac{1}{4}, \infty)$ and concave downward on $(-\infty, -\frac{1}{4})$. There is an inflection point at $(-\frac{1}{4}, f(-\frac{1}{4})) = (-\frac{1}{4}, \frac{21}{8})$.

10. (a) $f(x) = \dfrac{x^2}{x^2+3} \quad\Rightarrow\quad f'(x) = \dfrac{(x^2+3)(2x) - x^2(2x)}{(x^2+3)^2} = \dfrac{6x}{(x^2+3)^2}$. The denominator is positive so the sign of $f'(x)$ is determined by the sign of x. Thus, $f'(x) > 0 \;\Leftrightarrow\; x > 0$ and $f'(x) < 0 \;\Leftrightarrow\; x < 0$. So f is increasing on $(0, \infty)$ and f is decreasing on $(-\infty, 0)$.

(b) f changes from decreasing to increasing at $x = 0$. Thus, $f(0) = 0$ is a local minimum value.

(c) $f''(x) = \dfrac{(x^2+3)^2(6) - 6x \cdot 2(x^2+3)(2x)}{[(x^2+3)^2]^2} = \dfrac{6(x^2+3)[x^2+3-4x^2]}{(x^2+3)^4} = \dfrac{6(3-3x^2)}{(x^2+3)^3} = \dfrac{-18(x+1)(x-1)}{(x^2+3)^3}$.

$f''(x) > 0 \;\Leftrightarrow\; -1 < x < 1$ and $f''(x) < 0 \;\Leftrightarrow\; x < -1$ or $x > 1$. Thus, f is concave upward on $(-1, 1)$ and concave downward on $(-\infty, -1)$ and $(1, \infty)$. There are inflection points at $\left(\pm 1, \frac{1}{4}\right)$.

15. (a) $y = f(x) = \dfrac{\ln x}{\sqrt{x}}$. (Note that f is only defined for $x > 0$.)

$$f'(x) = \frac{\sqrt{x}\,(1/x) - \ln x\left(\frac{1}{2}x^{-1/2}\right)}{x} = \frac{\dfrac{1}{\sqrt{x}} - \dfrac{\ln x}{2\sqrt{x}}}{x} \cdot \frac{2\sqrt{x}}{2\sqrt{x}} = \frac{2 - \ln x}{2x^{3/2}} > 0 \;\Leftrightarrow\; 2 - \ln x > 0 \;\Leftrightarrow$$

$\ln x < 2 \;\Leftrightarrow\; x < e^2$. Therefore f is increasing on $(0, e^2)$ and decreasing on (e^2, ∞).

(b) f changes from increasing to decreasing at $x = e^2$, so $f(e^2) = \dfrac{\ln e^2}{\sqrt{e^2}} = \dfrac{2}{e}$ is a local maximum value.

(c) $f''(x) = \dfrac{2x^{3/2}(-1/x) - (2 - \ln x)(3x^{1/2})}{(2x^{3/2})^2} = \dfrac{-2x^{1/2} + 3x^{1/2}(\ln x - 2)}{4x^3} = \dfrac{x^{1/2}(-2 + 3\ln x - 6)}{4x^3} = \dfrac{3\ln x - 8}{4x^{5/2}}$

$f''(x) = 0 \;\Leftrightarrow\; \ln x = \frac{8}{3} \;\Leftrightarrow\; x = e^{8/3}$. $f''(x) > 0 \;\Leftrightarrow\; x > e^{8/3}$, so f is concave upward on $(e^{8/3}, \infty)$ and concave downward on $(0, e^{8/3})$. There is an inflection point at $\left(e^{8/3}, \frac{8}{3}e^{-4/3}\right) \approx (14.39, 0.70)$.

24. (a) $g(x) = 200 + 8x^3 + x^4 \quad\Rightarrow\quad g'(x) = 24x^2 + 4x^3 = 4x^2(6 + x) = 0$ when $x = -6$ and when $x = 0$.

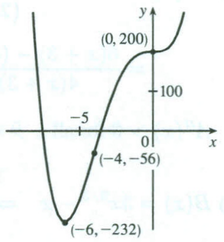
$g'(x) > 0 \;\Leftrightarrow\; x > -6 \;\; [x \neq 0]$ and $g'(x) < 0 \;\Leftrightarrow\; x < -6$, so g is decreasing on $(-\infty, -6)$ and g is increasing on $(-6, \infty)$, with a horizontal tangent at $x = 0$.

(b) $g(-6) = -232$ is a local minimum value. There is no local maximum value.

(c) $g''(x) = 48x + 12x^2 = 12x(4 + x) = 0$ when $x = -4$ and when $x = 0$.

$g''(x) > 0 \;\Leftrightarrow\; x < -4$ or $x > 0$ and $g''(x) < 0 \;\Leftrightarrow\; -4 < x < 0$, so g is CU on $(-\infty, -4)$ and $(0, \infty)$, and g is CD on $(-4, 0)$. There are inflection points at $(-4, -56)$ and $(0, 200)$.

(d)

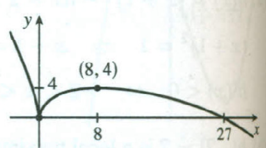


28. (a) $B(x) = 3x^{2/3} - x \quad\Rightarrow\quad B'(x) = 2x^{-1/3} - 1 = \dfrac{2}{\sqrt[3]{x}} - 1 = \dfrac{2 - \sqrt[3]{x}}{\sqrt[3]{x}}$. $B'(x) > 0$ if $0 < x < 8$ and $B'(x) < 0$ if $x < 0$ or $x > 8$, so B is decreasing on $(-\infty, 0)$ and $(8, \infty)$, and B is increasing on $(0, 8)$.

(b) $B(0) = 0$ is a local minimum value. $B(8) = 4$ is a local maximum value.

(c) $B''(x) = -\frac{2}{3}x^{-4/3} = \dfrac{-2}{3x^{4/3}}$, so $B''(x) < 0$ for all $x \neq 0$. B is concave downward on $(-\infty, 0)$ and $(0, \infty)$. There is no inflection point.

(d)



31. (a) $f(\theta) = 2\cos\theta + \cos^2\theta,\ 0 \le \theta \le 2\pi \quad \Rightarrow \quad f'(\theta) = -2\sin\theta + 2\cos\theta\,(-\sin\theta) = -2\sin\theta\,(1+\cos\theta)$.

$f'(\theta) = 0 \quad \Leftrightarrow \quad \theta = 0, \pi$, and 2π. $f'(\theta) > 0 \quad \Leftrightarrow \quad \pi < \theta < 2\pi$ and $f'(\theta) < 0 \quad \Leftrightarrow \quad 0 < \theta < \pi$. So f is increasing on $(\pi, 2\pi)$ and f is decreasing on $(0, \pi)$.

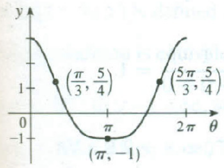
(b) $f(\pi) = -1$ is a local minimum value.

(c) $f'(\theta) = -2\sin\theta\,(1+\cos\theta) \quad \Rightarrow$

$$\begin{aligned} f''(\theta) &= -2\sin\theta\,(-\sin\theta) + (1+\cos\theta)(-2\cos\theta) = 2\sin^2\theta - 2\cos\theta - 2\cos^2\theta \\ &= 2(1-\cos^2\theta) - 2\cos\theta - 2\cos^2\theta = -4\cos^2\theta - 2\cos\theta + 2 \\ &= -2(2\cos^2\theta + \cos\theta - 1) = -2(2\cos\theta - 1)(\cos\theta + 1) \end{aligned}$$

Since $-2(\cos\theta + 1) < 0$ [for $\theta \ne \pi$], $f''(\theta) > 0 \quad \Rightarrow \quad 2\cos\theta - 1 < 0 \quad \Rightarrow \quad \cos\theta < \frac{1}{2} \quad \Rightarrow \quad \frac{\pi}{3} < \theta < \frac{5\pi}{3}$ and $f''(\theta) < 0 \quad \Rightarrow \quad \cos\theta > \frac{1}{2} \quad \Rightarrow \quad 0 < \theta < \frac{\pi}{3}$ or $\frac{5\pi}{3} < \theta < 2\pi$. So f is CU on $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$ and f is CD on $\left(0, \frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3}, 2\pi\right)$. There are points of inflection at $\left(\frac{\pi}{3}, f\left(\frac{\pi}{3}\right)\right) = \left(\frac{\pi}{3}, \frac{5}{4}\right)$ and $\left(\frac{5\pi}{3}, f\left(\frac{5\pi}{3}\right)\right) = \left(\frac{5\pi}{3}, \frac{5}{4}\right)$.

(d)



34. $f(x) = \dfrac{x^2}{(x-2)^2}$ has domain $(-\infty, 2) \cup (2, \infty)$.

(a) $\displaystyle\lim_{x\to\pm\infty} \frac{x^2}{x^2 - 4x + 4} = \lim_{x\to\pm\infty} \frac{x^2/x^2}{(x^2 - 4x + 4)/x^2} = \lim_{x\to\pm\infty} \frac{1}{1 - 4/x + 4/x^2} = \frac{1}{1 - 0 + 0} = 1$,

so $y = 1$ is a HA. $\displaystyle\lim_{x\to 2^+} \frac{x^2}{(x-2)^2} = \infty$ since $x^2 \to 4$ and $(x-2)^2 \to 0^+$ as $x \to 2^+$, so $x = 2$ is a VA.

(b) $f(x) = \dfrac{x^2}{(x-2)^2} \quad \Rightarrow \quad f'(x) = \dfrac{(x-2)^2(2x) - x^2 \cdot 2(x-2)}{[(x-2)^2]^2} = \dfrac{2x(x-2)[(x-2) - x]}{(x-2)^4} = \dfrac{-4x}{(x-2)^3}$.

$f'(x) > 0$ if $0 < x < 2$ and $f'(x) < 0$ if $x < 0$ or $x > 2$, so f is increasing on $(0, 2)$ and f is decreasing on $(-\infty, 0)$ and $(2, \infty)$.

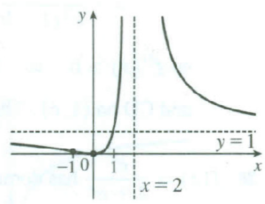
(c) $f(0) = 0$ is a local minimum value.

(d) $f''(x) = \dfrac{(x-2)^3(-4) - (-4x)\cdot 3(x-2)^2}{[(x-2)^3]^2}$

$= \dfrac{4(x-2)^2[-(x-2) + 3x]}{(x-2)^6} = \dfrac{8(x+1)}{(x-2)^4}$

$f''(x) > 0$ if $x > -1$ $(x \ne 2)$ and $f''(x) < 0$ if $x < -1$. Thus, f is CU on $(-1, 2)$ and $(2, \infty)$, and f is CD on $(-\infty, -1)$. There is an inflection point at $\left(-1, \frac{1}{9}\right)$.

(e)



38. $f(x) = \dfrac{e^x}{1+e^x}$ has domain $\mathbb{R}$.

(a) $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{e^x/e^x}{(1+e^x)/e^x} = \lim\limits_{x\to\infty} \dfrac{1}{e^{-x}+1} = \dfrac{1}{0+1} = 1$, so $y = 1$ is a HA.

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{e^x}{1+e^x} = \dfrac{0}{1+0} = 0$, so $y = 0$ is a HA. No VA.

(b) $f'(x) = \dfrac{(1+e^x)e^x - e^x \cdot e^x}{(1+e^x)^2} = \dfrac{e^x}{(1+e^x)^2} > 0$ for all x. Thus, f is increasing on $\mathbb{R}$.

(c) There is no local maximum or minimum.

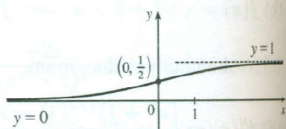
(d) $f''(x) = \dfrac{(1+e^x)^2 e^x - e^x \cdot 2(1+e^x)e^x}{[(1+e^x)^2]^2}$

$= \dfrac{e^x(1+e^x)[(1+e^x) - 2e^x]}{(1+e^x)^4} = \dfrac{e^x(1-e^x)}{(1+e^x)^3}$

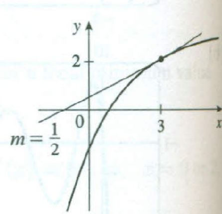
$f''(x) > 0 \iff 1 - e^x > 0 \iff x < 0$, so f is CU on $(-\infty, 0)$ and CD on $(0, \infty)$.

There is an inflection point at $\left(0, \tfrac{1}{2}\right)$.

(e)



50. (a) $f(3) = 2 \Rightarrow$ the point $(3, 2)$ is on the graph of f. $f'(3) = \frac{1}{2} \Rightarrow$ the slope of the tangent line at $(3, 2)$ is $\frac{1}{2}$. $f'(x) > 0$ for all $x \Rightarrow f$ is increasing on $\mathbb{R}$. $f''(x) < 0$ for all $x \Rightarrow f$ is concave downward on $\mathbb{R}$. A possible graph for f is shown.

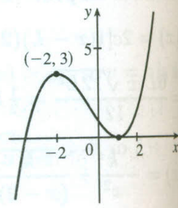


(b) The tangent line at $(3, 2)$ has equation $y - 2 = \frac{1}{2}(x - 3)$, or $y = \frac{1}{2}x + \frac{1}{2}$, and x-intercept -1. Since f is concave downward on $\mathbb{R}$, f is below the x-axis at $x = -1$, and hence changes sign at least once. Since f is increasing on $\mathbb{R}$, it changes sign at most once. Thus, it changes sign exactly once and there is one solution of the equation $f(x) = 0$.

(c) $f'' < 0 \Rightarrow f'$ is decreasing. Since $f'(3) = \frac{1}{2}$, $f'(2)$ must be greater than $\frac{1}{2}$, so no, it is not possible that $f'(2) = \frac{1}{3}$.

59. $f(x) = ax^3 + bx^2 + cx + d \Rightarrow f'(x) = 3ax^2 + 2bx + c$.

We are given that $f(1) = 0$ and $f(-2) = 3$, so $f(1) = a + b + c + d = 0$ and $f(-2) = -8a + 4b - 2c + d = 3$. Also $f'(1) = 3a + 2b + c = 0$ and $f'(-2) = 12a - 4b + c = 0$ by Fermat's Theorem. Solving these four equations, we get $a = \frac{2}{9}$, $b = \frac{1}{3}$, $c = -\frac{4}{3}$, $d = \frac{7}{9}$, so the function is $f(x) = \frac{1}{9}(2x^3 + 3x^2 - 12x + 7)$.



61. $f(x) = \tan x - x \Rightarrow f'(x) = \sec^2 x - 1 > 0$ for $0 < x < \frac{\pi}{2}$ since $\sec^2 x > 1$ for $0 < x < \frac{\pi}{2}$. So f is increasing on $\left(0, \frac{\pi}{2}\right)$. Thus, $f(x) > f(0) = 0$ for $0 < x < \frac{\pi}{2} \Rightarrow \tan x - x > 0 \Rightarrow \tan x > x$ for $0 < x < \frac{\pi}{2}$.

64. If $3 \le f'(x) \le 5$ for all x, then by the Mean Value Theorem, $f(8) - f(2) = f'(c) \cdot (8 - 2)$ for some c in $[2, 8]$. (f is differentiable for all x, so, in particular, f is differentiable on $(2, 8)$ and continuous on $[2, 8]$. Thus, the hypotheses of the Mean Value Theorem are satisfied.) Since $f(8) - f(2) = 6f'(c)$ and $3 \le f'(c) \le 5$, it follows that

$6 \cdot 3 \le 6f'(c) \le 6 \cdot 5 \Rightarrow 18 \le f(8) - f(2) \le 30$.