

Homework 9 – Solutions

Section 4.5

3. (a) When x is near a , $f(x)$ is near 0 and $p(x)$ is large, so $f(x) - p(x)$ is large negative. Thus, $\lim_{x \rightarrow a} [f(x) - p(x)] = -\infty$.
- (b) $\lim_{x \rightarrow a} [p(x) - q(x)]$ is an indeterminate form of type $\infty - \infty$.
- (c) When x is near a , $p(x)$ and $q(x)$ are both large, so $p(x) + q(x)$ is large. Thus, $\lim_{x \rightarrow a} [p(x) + q(x)] = \infty$.
4. (a) $\lim_{x \rightarrow a} [f(x)]^{g(x)}$ is an indeterminate form of type 0^0 .
- (b) If $y = [f(x)]^{p(x)}$, then $\ln y = p(x) \ln f(x)$. When x is near a , $p(x) \rightarrow \infty$ and $\ln f(x) \rightarrow -\infty$, so $\ln y \rightarrow -\infty$.
Therefore, $\lim_{x \rightarrow a} [f(x)]^{p(x)} = \lim_{x \rightarrow a} y = \lim_{x \rightarrow a} e^{\ln y} = 0$, provided f^p is defined.
- (c) $\lim_{x \rightarrow a} [h(x)]^{p(x)}$ is an indeterminate form of type 1^∞ .
- (d) $\lim_{x \rightarrow a} [p(x)]^{f(x)}$ is an indeterminate form of type ∞^0 .
- (e) If $y = [p(x)]^{q(x)}$, then $\ln y = q(x) \ln p(x)$. When x is near a , $q(x) \rightarrow \infty$ and $\ln p(x) \rightarrow \infty$, so $\ln y \rightarrow \infty$. Therefore,
 $\lim_{x \rightarrow a} [p(x)]^{q(x)} = \lim_{x \rightarrow a} y = \lim_{x \rightarrow a} e^{\ln y} = \infty$.
- (f) $\lim_{x \rightarrow a} \sqrt[q(x)]{p(x)} = \lim_{x \rightarrow a} [p(x)]^{1/q(x)}$ is an indeterminate form of type ∞^0 .

8. This limit has the form $\frac{0}{0}$. $\lim_{x \rightarrow 0} \frac{\sin 4x}{\tan 5x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{4 \cos 4x}{5 \sec^2(5x)} = \frac{4(1)}{5(1)^2} = \frac{4}{5}$

12. $\lim_{\theta \rightarrow \pi/2} \frac{1 - \sin \theta}{\csc \theta} = \frac{0}{1} = 0$. L'Hospital's Rule does not apply.

14. This limit has the form $\frac{\infty}{\infty}$. $\lim_{x \rightarrow \infty} \frac{(\ln x)^2}{x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{2(\ln x)(1/x)}{1} = 2 \lim_{x \rightarrow \infty} \frac{\ln x}{x} \stackrel{H}{=} 2 \lim_{x \rightarrow \infty} \frac{1/x}{1} = 2(0) = 0$

28. This limit has the form $\infty \cdot 0$. $\lim_{x \rightarrow -\infty} x^2 e^x = \lim_{x \rightarrow -\infty} \frac{x^2}{e^{-x}} \stackrel{H}{=} \lim_{x \rightarrow -\infty} \frac{2x}{-e^{-x}} \stackrel{H}{=} \lim_{x \rightarrow -\infty} \frac{2}{e^{-x}} = \lim_{x \rightarrow -\infty} 2e^x = 0$

32. This limit has the form $\infty \cdot 0$.

$$\lim_{x \rightarrow \infty} x \tan(1/x) = \lim_{x \rightarrow \infty} \frac{\tan(1/x)}{1/x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{\sec^2(1/x)(-1/x^2)}{-1/x^2} = \lim_{x \rightarrow \infty} \sec^2(1/x) = 1^2 = 1$$

38. As $x \rightarrow \infty$, $1/x \rightarrow 0$, and $e^{1/x} \rightarrow 1$. So the limit has the form $\infty - \infty$ and we will change the form to a product by factoring out x .

$$\lim_{x \rightarrow \infty} (xe^{1/x} - x) = \lim_{x \rightarrow \infty} x(e^{1/x} - 1) = \lim_{x \rightarrow \infty} \frac{e^{1/x} - 1}{1/x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^{1/x}(-1/x^2)}{-1/x^2} = \lim_{x \rightarrow \infty} e^{1/x} = e^0 = 1$$

40. $y = (\tan 2x)^x \Rightarrow \ln y = x \cdot \ln \tan 2x$, so

$$\begin{aligned} \lim_{x \rightarrow 0^+} \ln y &= \lim_{x \rightarrow 0^+} x \cdot \ln \tan 2x = \lim_{x \rightarrow 0^+} \frac{\ln \tan 2x}{1/x} \stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{(1/\tan 2x)(2 \sec^2 2x)}{-1/x^2} = \lim_{x \rightarrow 0^+} \frac{-2x^2 \cos 2x}{\sin 2x \cos^2 2x} \\ &= \lim_{x \rightarrow 0^+} \frac{2x}{\sin 2x} \cdot \lim_{x \rightarrow 0^+} \frac{-x}{\cos 2x} = 1 \cdot 0 = 0 \Rightarrow \\ \lim_{x \rightarrow 0^+} (\tan 2x)^x &= \lim_{x \rightarrow 0^+} e^{\ln y} = e^0 = 1. \end{aligned}$$

44. $y = x^{(\ln 2)/(1 + \ln x)} \Rightarrow \ln y = \frac{\ln 2}{1 + \ln x} \ln x \Rightarrow$

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{(\ln 2)(\ln x)}{1 + \ln x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{(\ln 2)(1/x)}{1/x} = \lim_{x \rightarrow \infty} \ln 2 = \ln 2, \text{ so } \lim_{x \rightarrow \infty} x^{(\ln 2)/(1 + \ln x)} = \lim_{x \rightarrow \infty} e^{\ln y} = e^{\ln 2} = 2.$$

46. $y = (2 - x)^{\tan(\pi x/2)} \Rightarrow \ln y = \tan\left(\frac{\pi x}{2}\right) \ln(2 - x) \Rightarrow$

$$\begin{aligned} \lim_{x \rightarrow 1} \ln y &= \lim_{x \rightarrow 1} \left[\tan\left(\frac{\pi x}{2}\right) \ln(2 - x) \right] = \lim_{x \rightarrow 1} \frac{\ln(2 - x)}{\cot\left(\frac{\pi x}{2}\right)} \stackrel{H}{=} \lim_{x \rightarrow 1} \frac{\frac{1}{2-x}(-1)}{-\csc^2\left(\frac{\pi x}{2}\right) \cdot \frac{\pi}{2}} = \frac{2}{\pi} \lim_{x \rightarrow 1} \frac{\sin^2\left(\frac{\pi x}{2}\right)}{2 - x} \\ &= \frac{2}{\pi} \cdot \frac{1^2}{1} = \frac{2}{\pi} \Rightarrow \lim_{x \rightarrow 1} (2 - x)^{\tan(\pi x/2)} = \lim_{x \rightarrow 1} e^{\ln y} = e^{(2/\pi)} \end{aligned}$$

73.

Since $f(2) = 0$, the given limit has the form $\frac{0}{0}$.

$$\lim_{x \rightarrow 0} \frac{f(2 + 3x) + f(2 + 5x)}{x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{f'(2 + 3x) \cdot 3 + f'(2 + 5x) \cdot 5}{1} = f'(2) \cdot 3 + f'(2) \cdot 5 = 8f'(2) = 8 \cdot 7 = 56$$

Section 4.6

4. Call the two numbers x and y . Then $x + y = 16$, so $y = 16 - x$. Call the sum of their squares S . Then

$$S = x^2 + y^2 = x^2 + (16 - x)^2 \Rightarrow S' = 2x + 2(16 - x)(-1) = 2x - 32 + 2x = 4x - 32. S' = 0 \Rightarrow x = 8.$$

Since $S'(x) < 0$ for $0 < x < 8$ and $S'(x) > 0$ for $x > 8$, there is an absolute minimum at $x = 8$. Thus, $y = 16 - 8 = 8$

and $S = 8^2 + 8^2 = 128$.

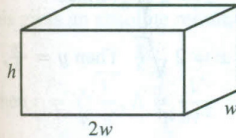
12. Let b be the length of the base of the box and h the height. The volume is $32,000 = b^2 h \Rightarrow h = 32,000/b^2$.

The surface area of the open box is $S = b^2 + 4hb = b^2 + 4(32,000/b^2)b = b^2 + 4(32,000)/b$.

So $S'(b) = 2b - 4(32,000)/b^2 = 2(b^3 - 64,000)/b^2 = 0 \Leftrightarrow b = \sqrt[3]{64,000} = 40$. This gives an absolute minimum

since $S'(b) < 0$ if $0 < b < 40$ and $S'(b) > 0$ if $b > 40$. The box should be $40 \times 40 \times 20$.

14.



$$V = lwh \Rightarrow 10 = (2w)(w)h = 2w^2h, \text{ so } h = 5/w^2.$$

The cost is $10(2w^2) + 6[2(2wh) + 2(hw)] = 20w^2 + 36wh$, so

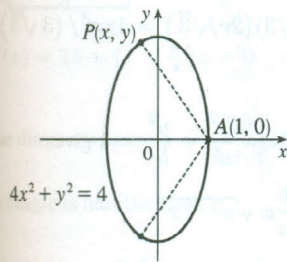
$$C(w) = 20w^2 + 36w(5/w^2) = 20w^2 + 180/w.$$

$C'(w) = 40w - 180/w^2 = 40(w^3 - \frac{9}{2})/w^2 \Rightarrow w = \sqrt[3]{\frac{9}{2}}$ is the critical number. There is an absolute minimum for C

when $w = \sqrt[3]{\frac{9}{2}}$ since $C'(w) < 0$ for $0 < w < \sqrt[3]{\frac{9}{2}}$ and $C'(w) > 0$ for $w > \sqrt[3]{\frac{9}{2}}$.

$$C\left(\sqrt[3]{\frac{9}{2}}\right) = 20\left(\sqrt[3]{\frac{9}{2}}\right)^2 + \frac{180}{\sqrt[3]{9/2}} \approx \$163.54.$$

15.



From the figure, we see that there are two points that are farthest away from $A(1, 0)$. The distance d from A to an arbitrary point $P(x, y)$ on the ellipse is

$$d = \sqrt{(x-1)^2 + (y-0)^2} \text{ and the square of the distance is}$$

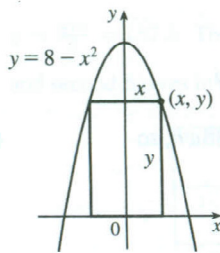
$$S = d^2 = x^2 - 2x + 1 + y^2 = x^2 - 2x + 1 + (4 - 4x^2) = -3x^2 - 2x + 5.$$

$S' = -6x - 2$ and $S' = 0 \Rightarrow x = -\frac{1}{3}$. Now $S'' = -6 < 0$, so we know that S has a maximum at $x = -\frac{1}{3}$. Since $-1 \leq x \leq 1$, $S(-1) = 4$,

$S(-\frac{1}{3}) = \frac{16}{3}$, and $S(1) = 0$, we see that the maximum distance is $\sqrt{\frac{16}{3}}$. The corresponding y -values are

$$y = \pm\sqrt{4 - 4(-\frac{1}{3})^2} = \pm\sqrt{\frac{32}{9}} = \pm\frac{4}{3}\sqrt{2} \approx \pm 1.89. \text{ The points are } (-\frac{1}{3}, \pm\frac{4}{3}\sqrt{2}).$$

18.



The rectangle has area $A(x) = 2xy = 2x(8 - x^2) = 16x - 2x^3$, where

$$0 \leq x \leq 2\sqrt{2}. \text{ Now } A'(x) = 16 - 6x^2 = 0 \Rightarrow x = 2\sqrt{\frac{2}{3}}. \text{ Since}$$

$$A(0) = A(2\sqrt{2}) = 0, \text{ there is a maximum when } x = 2\sqrt{\frac{2}{3}}. \text{ Then } y = \frac{16}{3},$$

so the rectangle has dimensions $4\sqrt{\frac{2}{3}}$ and $\frac{16}{3}$.

40.

$$y = 1 + 40x^3 - 3x^5 \Rightarrow y' = 120x^2 - 15x^4, \text{ so the tangent line to the curve at } x = a \text{ has slope } m(a) = 120a^2 - 15a^4.$$

Now $m'(a) = 240a - 60a^3 = -60a(a^2 - 4) = -60a(a+2)(a-2)$, so $m'(a) > 0$ for $a < -2$, and $0 < a < 2$, and

$m'(a) < 0$ for $-2 < a < 0$ and $a > 2$. Thus, m is increasing on $(-\infty, -2)$, decreasing on $(-2, 0)$, increasing on $(0, 2)$, and

decreasing on $(2, \infty)$. Clearly, $m(a) \rightarrow -\infty$ as $a \rightarrow \pm\infty$, so the maximum value of $m(a)$ must be one of the two local

maxima, $m(-2)$ or $m(2)$. But both $m(-2)$ and $m(2)$ equal $120 \cdot 2^2 - 15 \cdot 2^4 = 480 - 240 = 240$. So 240 is the largest

slope, and it occurs at the points $(-2, -223)$ and $(2, 225)$. Note: $a = 0$ corresponds to a local minimum of m .

42. $y = 4 - x^2 \Rightarrow y' = -2x$, so an equation of the tangent line at $(a, 4 - a^2)$ is

$y - (4 - a^2) = -2a(x - a)$, or $y = -2ax + a^2 + 4$. The y -intercept $[x = 0]$

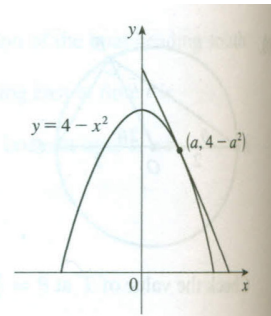
is $a^2 + 4$. The x -intercept $[y = 0]$ is $\frac{a^2 + 4}{2a}$. The area A of the triangle is

$$A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2} \cdot \frac{a^2 + 4}{2a} (a^2 + 4) = \frac{1}{4} \frac{a^4 + 8a^2 + 16}{a} = \frac{1}{4} \left(a^3 + 8a + \frac{16}{a} \right).$$

$$A' = 0 \Rightarrow \frac{1}{4} \left(3a^2 + 8 - \frac{16}{a^2} \right) = 0 \Rightarrow 3a^4 + 8a^2 - 16 = 0 \Rightarrow$$

$$(3a^2 - 4)(a^2 + 4) = 0 \Rightarrow a^2 = \frac{4}{3} \Rightarrow a = \frac{2}{\sqrt{3}}. A'' = \frac{1}{4} \left(6a + \frac{32}{a^3} \right) > 0, \text{ so there is an absolute minimum at}$$

$$a = \frac{2}{\sqrt{3}}. \text{ Thus, } A = \frac{1}{2} \cdot \frac{4/3 + 4}{2(2/\sqrt{3})} \left(\frac{4}{3} + 4 \right) = \frac{1}{2} \cdot \frac{4\sqrt{3}}{3} \cdot \frac{16}{3} = \frac{32}{9} \sqrt{3}.$$



46.

(a) Let $p(x)$ be the demand function. Then $p(x)$ is linear and $y = p(x)$ passes through $(20, 10)$ and $(18, 11)$, so the slope is $-\frac{1}{2}$ and an equation of the line is $y - 10 = -\frac{1}{2}(x - 20) \Leftrightarrow y = -\frac{1}{2}x + 20$. Thus, the demand is $p(x) = -\frac{1}{2}x + 20$ and the revenue is $R(x) = xp(x) = -\frac{1}{2}x^2 + 20x$.

(b) The cost is $C(x) = 6x$, so the profit is $P(x) = R(x) - C(x) = -\frac{1}{2}x^2 + 14x$. Then $0 = P'(x) = -x + 14 \Rightarrow x = 14$. Since $P''(x) = -1 < 0$, the selling price for maximum profit is $p(14) = -\frac{1}{2}(14) + 20 = \13 .

48.

Let x denote the number of \$10 increases in rent. Then the price is $p(x) = 800 + 10x$, and the number of units occupied is $100 - x$. Now the revenue is

$$\begin{aligned} R(x) &= (\text{rental price per unit}) \times (\text{number of units rented}) \\ &= (800 + 10x)(100 - x) = -10x^2 + 200x + 80,000 \text{ for } 0 \leq x \leq 100 \Rightarrow \end{aligned}$$

$R'(x) = -20x + 200 = 0 \Leftrightarrow x = 10$. This is a maximum since $R''(x) = -20 < 0$ for all x . Now we must check the value of $R(x) = (800 + 10x)(100 - x)$ at $x = 10$ and at the endpoints of the domain to see which value of x gives the maximum value of R . $R(0) = 80,000$, $R(10) = (900)(90) = 81,000$, and $R(100) = (1800)(0) = 0$. Thus, the maximum revenue of \$81,000/week occurs when 90 units are occupied at a rent of \$900/week.