KOÇ UNIVERSITY

FALL 2016 MATH 102

Final January 15, 2017

Duration of Exam: 105 minutes

INSTRUCTIONS: Calculators are not allowed. No books, no notes, and talking allowed. You must always **explain your answers** and **show your work** to receive **full credit.** Use the back of these pages if necessary. **Print (use CAPITAL LETTERS) and sign your name, and indicate your section below.**

Name, Surname: KEY

Signature:

Section (Check One):

Section 1: A. Erdoğan (08:30. Tu. Th.)
Section 2: A. Erdoğan (10:00. Tu. Th.)

PROBLEM	POINTS	SCORE
1	15	
2	15	
3	40	
4	15	
5	15	
6	10	
TOTAL	**110 (10 pts. bonus)**	

1. (15 points) Find the absolute maximum and minimum of $f(x) = \frac{(x^2-1)^2}{2} - x^2 + 1$ on the interval $[-1, 2]$.

$f(x)$ is continous on $[-1, 2]$.

$f'(x) = (x^2-1) \cdot 2x - 2x = 2x(x^2-2) = 0$

$\Rightarrow$ $x = 0$ and $x = \pm\sqrt{2}$. But $-\sqrt{2} \notin [-1, 2]$

so the only critical numbers of $f(x)$ on $[-1, 2]$ are

$x = 0$ and $x = \sqrt{2}$.

$f(0) = \frac{1}{2} + 1 = \frac{3}{2}$

$f(\sqrt{2}) = \frac{1}{2} - 2 + 1 = \frac{-1}{2}$

$f(-1) = -1 + 1 = 0$

$f(2) = \frac{9}{2} - 4 + 1 = \frac{3}{2}$

So the absolute max. of $f(x)$ on $[-1, 2]$ is $f(0) = f(2) = \frac{3}{2}$ //

and the absolute min. of $f(x)$ on $[-1, 2]$ is $f(\sqrt{2}) = \frac{-1}{2}$ //

2. (15 points) Show that $f(x) = x^3 + \sin(x)$ has exactly one real root on the interval $[\frac{-\pi}{2}, \frac{\pi}{2}]$.

$f(x)$ is diffferentiable on $(\frac{-\pi}{2}, \frac{\pi}{2})$ and

is continous on $[\frac{-\pi}{2}, \frac{\pi}{2}]$.

$f(\frac{-\pi}{2}) = \frac{-\pi^3}{8} - 1 < 0$, $f(\frac{\pi}{2}) = \frac{\pi^3}{8} + 1 > 0$

So by Intermediate value theorem $f(x)$ has a root $c \in (\frac{-\pi}{2}, \frac{\pi}{2})$.

Now suppose that $f(x)$ has two roots $c, d \in (\frac{-\pi}{2}, \frac{\pi}{2})$.

Then $f(c) = f(d) = 0$, so by Rolle's theorem (or mean value theorem)

there exists $r \in (c, d)$ such that

$$f'(r) = \frac{f(c) - f(d)}{c - d} = 0.$$

But $f'(r) = 3r^2 + \cos(r) > 0$ for any $r \in (\frac{-\pi}{2}, \frac{\pi}{2})$

Thus $f(x)$ has exactly one real root on $[\frac{-\pi}{2}, \frac{\pi}{2}]$ //

3. (40 points) Evaluate the following integrals.

(a) (10 points) $\int x\cos^2(x)\,dx$ $= \int x\cdot\left(\frac{1+\cos(2x)}{2}\right)dx$

$$= \int \frac{x}{2}\,dx + \int \frac{x\cdot\cos(2x)}{2}\,dx = \frac{x^2}{4} + \frac{x\sin(2x)}{4} - \frac{1}{4}\int \sin(2x)\,dx$$

$$\left(\begin{array}{l} u = x, \quad dv = \cos(2x)\,dx \\ du = dx, \quad v = \frac{\sin(2x)}{2} \end{array}\right)$$

$$= \frac{x^2}{4} + \frac{x\sin(2x)}{4} + \frac{\cos(2x)}{8} + C \quad //$$

(b) (10 points) $\int \frac{x^2-x+8}{x^3+4x}\,dx$ $= \int\left(\frac{2}{x} - \frac{x}{x^2+4} - \frac{1}{x^2+4}\right)dx$

$$\frac{x^2-x+8}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$\Rightarrow A(x^2+4) + (Bx+C)x = x^2 - x + 8$$

$$\Rightarrow A = 2,\ C = -1,\ B = -1$$

$$= \int \frac{2}{x}\,dx - \int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx$$

$$\left(\begin{array}{l} u = x^2+4 \\ du = 2x\,dx \end{array}\right)$$

$$= 2\ln|x| - \frac{\ln(x^2+4)}{2} - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C \quad //$$

(c) (8 points) $\displaystyle\int_{-\pi}^{\pi} \frac{x^2 \sin(x)}{x^4+1}\,dx$

x^2 and x^4+1 are even,

and $\sin(x)$ is odd. So

$\dfrac{x^2\sin(x)}{x^4+1}$ is an odd function

and continous on $[-\pi, \pi]$

$$\Rightarrow \int_{-\pi}^{\pi} \frac{x^2\sin(x)}{x^4+1}\,dx = 0 \,//$$

(d) (12 points) $\displaystyle\int_{e}^{\infty} \frac{1}{x(\ln x)^3}\,dx = \lim_{R\to\infty} \int_{e}^{R} \frac{1}{x.(\ln x)^3}\,dx$

$\left(u = \ln x,\ du = \frac{1}{x}dx \right)$

$$= \lim_{R\to\infty} \int_{1}^{\ln(R)} \frac{1}{u^3}\,du$$

$$= \lim_{R\to\infty} \left. \frac{-1}{2u^2} \right|_{1}^{\ln(R)} = \lim_{R\to\infty} \left(\frac{1}{2} - \frac{1}{2.\ln(R)^2} \right) = \frac{1}{2} \,//$$

$\left(\text{since } \lim_{R\to\infty} (\ln(R)) = \infty \right)$

4. (15 points) Determine whether the following integral is convergent or not [redacted].

$$\int_{-1}^{1} \frac{e^x}{e^x - 1}\,dx = \int_{-1}^{0} \frac{e^x}{e^x-1}\,dx + \int_{0}^{1} \frac{e^x}{e^x-1}\,dx$$

$$\int_{0}^{1} \frac{e^x}{e^x-1}\,dx = \lim_{t \to 0^+} \int_{t}^{1} \frac{e^x}{e^x-1}\,dx = \lim_{t \to 0^+} \int_{e^t-1}^{e-1} \frac{1}{u}\,du$$

$$(u = e^x - 1,\ du = e^x dx)$$

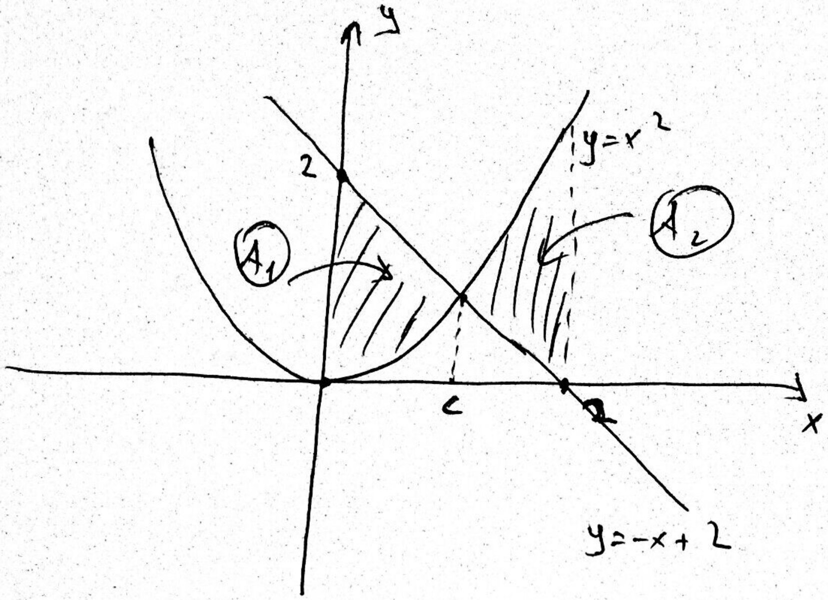
$$= \lim_{t\to 0^+} \ln|u| \Big|_{e^t-1}^{e-1} = \lim_{t \to 0^+} \left[\ln(e-1) - \ln(e^t-1)\right] = \infty$$

$$\left(\text{since } \lim_{t\to 0^+} (e^t - 1) = 0\right)$$

So $\int_{0}^{1} \frac{e^x}{e^x-1}\,dx$ is not convergent

$\Rightarrow \int_{-1}^{1} \frac{e^x}{e^x-1}\,dx$ is not convergent //

5. (15 points) Find the area of the region enclosed by the parabola $y = x^2$ and the lines $y = -x + 2$, [redacted] $x = 0$ and $x = 2$.



$y = x^2 = -x + 2$

$\Rightarrow \; x^2 + x - 2 = 0$

$\Rightarrow \; (x+2)(x-1) = 0$

$\Rightarrow \; x = -2 \quad \text{or} \quad x = 1$

$\Rightarrow \; c = 1$ //

$$\Rightarrow \; A_1 + A_2 = \int_0^1 (-x + 2 - x^2)\,dx + \int_1^2 \left[x^2 - (-x+2)\right] dx$$

$$= \left(\frac{-x^3}{3} - \frac{x^2}{2} + 2x\right)\Bigg|_0^1 + \left(\frac{x^3}{3} + \frac{x^2}{2} - 2x\right)\Bigg|_1^2$$

$$= \frac{-1}{3} - \frac{1}{2} + 2 + \frac{8}{3} + 2 - 4 - \frac{1}{3} - \frac{1}{2} + 2$$

$= 3$ //

6. (10 points) Let $f(x) = x^2 + x + b$ where $b \in \mathbb{R}$ is constant. Suppose that the line $y = 2x+1$ is the tangent line to the curve $y = f(x)$ at some point on the curve. Find b.

Let $P = (x_0, y_0)$ be the point on $y = f(x)$ at which the tangent line is $y = 2x+1$

$\Rightarrow \quad \underset{\text{(the slope)}}{2 = m} = f'(x_0) = 2x_0 + 1 \quad \Rightarrow \quad x_0 = \frac{1}{2}$

$\Rightarrow \quad y_0 = 2 \cdot x_0 + 1 = 2 \quad \Rightarrow \quad P = (x_0, y_0) = \left(\frac{1}{2}, 2\right)$ is on $y = f(x)$

$\Rightarrow \quad 2 = \left(\frac{1}{2}\right)^2 + \frac{1}{2} + b \quad \Rightarrow \quad b = 2 - \frac{1}{2} - \frac{1}{4}$

$\Rightarrow \quad b = \frac{5}{4}$ //