KOÇ UNIVERSITY

FALL 2016 MATH 102

Midterm I November 12, 2016

Duration of Exam: 75 minutes

INSTRUCTIONS: Calculators are not allowed. No books, no notes, and talking allowed. You must always **explain your answers** and **show your work** to receive **full credit**. Use the back of these pages if necessary. **Print (use CAPITAL LETTERS) and sign your name, and indicate your section below.**

Name, Surname: KEY

Signature: ____________________

Section (Check One):

Section 1: A. Erdoğan (08:30, Tu. Th.) —
Section 2: A. Erdoğan (10:00, Tu. Th.) —

PROBLEM	POINTS	SCORE
1	25	
2	10	
3	15	
4	30	
5	20	
TOTAL	100	

1. Find the following limits if they exist, otherwise state that they do not exist. Specify any infinite limits.

(a) (6 points) $\lim_{x\to-\infty} \frac{|x^3-1|}{2x^3+x-1}$ $= \lim_{x\to-\infty} \frac{-x^3+1}{2x^3+x-1}$ (Note: $x^3-1<0$)

$= \lim_{x\to-\infty} \frac{\cancel{x^3}\left(-1+\frac{1}{x^3}\right)}{\cancel{x^3}\left(2+\frac{1}{x^2}-\frac{1}{x^3}\right)} = \frac{-1}{2}$ //

(b) (6 points) $\lim_{x\to 4} \arcsin\left(\frac{x^2-5x+4}{x^2-2x-8}\right)$ $= \lim_{x\to 4} \arcsin\left(\frac{\cancel{(x-4)}(x-1)}{\cancel{(x-4)}(x+2)}\right)$

$= \arcsin\left(\lim_{x\to 4} \frac{x-1}{x+2}\right) = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$ //

(Note: arcsin(x) is continous)

(c) (6 points) $\lim_{x\to\pi/4} \frac{\tan x - 1}{x-\frac{\pi}{4}}$ (Hint: Express the limit as a derivative)

$\lim_{x\to\pi/4} \frac{\tan x-1}{x-\frac{\pi}{4}} = \lim_{x\to\frac{\pi}{4}} \frac{\tan x - \tan(\pi/4)}{x-\frac{\pi}{4}} = \left.\frac{d(\tan x)}{dx}\right|_{x=\frac{\pi}{4}} = \sec^2\left(\frac{\pi}{4}\right) = 2$ //

(d) (7 points) $\lim_{x\to 0^+} \left[(e^x-1)\left(\cos\left(\frac{\pi}{2x}\right)\right)\right]$

$-1 \leq \cos\left(\frac{\pi}{2x}\right) \leq 1$ for all $x \neq 0$. Also $e^x - 1 > 0$ if $x > 0$.

So $-(e^x-1) \leq (e^x-1)\cos\left(\frac{\pi}{2x}\right) \leq e^x - 1$.

Now $\lim_{x\to 0^+} -(e^x-1) = \lim_{x\to 0^+} (e^x-1) = 0$, thus by squeeze theorem

$\lim_{x\to 0^+} (e^x-1)\cdot\cos\left(\frac{\pi}{2x}\right) = 0$ //

2. (10 points) Find all values of a for which $f(x)$ is continuous where

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x}, & x < 0 \\ |x+a|, & x \geq 0 \end{cases}$$

Since $f(x)$ is continous, we have that

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

- $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sin x}{x} = 1$
- $f(0) = |a|$
- $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} |x+a| = |a|$

So $|a| = 1$

$\Rightarrow a = 1$ // or $a = -1$ //

3. (15 points) Show that the equation $1+\cos(x)-2^{\sin(x)}=0$ has a solution in the interval $[0,\pi/2]$.

Let $f(x)=1+\cos(x)-2^{\sin(x)}$, so that $f(x)$ is continous.

$\cdot\ f(0)=1+1-1=1>0$

$\cdot\ f\left(\frac{\pi}{2}\right)=1+0-2=-1<0$

So by Intermediate value theorem, there exists

$c\in\left[0,\frac{\pi}{2}\right]$ such that $f(c)=1+\cos(c)-2^{\sin(c)}=0$ //

4. Compute the derivative of $f(x)$ in each part.

(a) (8 points) $f(x) = \sin(e^{\sqrt{x}})$

$$f'(x) = \cos(e^{\sqrt{x}}) \cdot e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}} \cdot \cos(e^{\sqrt{x}})}{2\sqrt{x}} \quad //$$

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(b) (10 points) $f(x) = \arctan\left(\frac{2x^2-1}{x+1}\right)$ (Note: $\arctan(x)$ is also denoted by $\tan^{-1} x$).

$$f'(x) = \frac{1}{1+\left(\frac{2x^2-1}{x+1}\right)^2} \cdot \frac{d}{dx}\left(\frac{2x^2-1}{x+1}\right)$$

$$= \frac{\cancel{(x+1)^2}}{(x+1)^2+(2x^2-1)^2} \cdot \left(\frac{4x(x+1) - 2x^2+1}{\cancel{(x+1)^2}}\right) = \frac{2x^2+4x+1}{4x^4-3x^2+2x+2} \quad //$$

(c) (12 points) $f(x) = (\tan x)^x$, $x \in (0, \pi/2)$ (Hint: Use logarithmic differentiation).

$$\ln f(x) = x \cdot \ln(\tan x) \Rightarrow \frac{f'(x)}{f(x)} = \ln(\tan x) + x \cdot \frac{1}{\tan x} \cdot \sec^2 x$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \ln(\tan x) + x \cdot \frac{1}{\sin(x) \cdot \cos(x)} = \ln(\tan x) + \frac{2x}{\sin(2x)}$$

$$\Rightarrow f'(x) = (\tan x)^x \cdot \left[\ln(\tan x) + \frac{2x}{\sin(2x)}\right] \quad //$$

5. (20 points) Find an equation of the tangent line to the curve

$$xy^2 + \ln(x^2 + y) = 1$$

at the point $(0, e)$.

$$\frac{d}{dx}\left[xy^2 + \ln(x^2+y)\right] = \frac{d}{dx}(1)$$

$$\Rightarrow y^2 + 2xy \cdot y' + \frac{1}{x^2+y} \cdot (2x + y') = 0$$

Put $(x,y) = (0,e)$ to find the slope of the tangent line

$$\Rightarrow e^2 + \frac{1}{e} \cdot y'\Big|_{(0,e)} = 0 \Rightarrow \text{the slope is}$$

$$m = y'\Big|_{(0,e)} = -e^3.$$

So an equation is of the form $y = -e^3x + n$

Put $(x,y) = (0,e) \Rightarrow n = e$

Thus an equation of the tangent line is

$$y = -e^3x + e$$