KOÇ UNIVERSITY

FALL 2016 MATH 102

Midterm II December 17, 2016

Duration of Exam: 90 minutes

INSTRUCTIONS: Calculators are not allowed. No books, no notes, and talking allowed. You must always **explain your answers** and **show your work** to receive **full credit**. Use the back of these pages if necessary. **Print (use CAPITAL LETTERS) and sign your name, and indicate your section below.**

Name, Surname: KEY

Signature: ______

Section (Check One):

Section 1: A. Erdoğan (08:30, Tu. Th.) ___
Section 2: A. Erdoğan (10:00, Tu. Th.) ___

PROBLEM	POINTS	SCORE
1	8	
2	7	
3	15	
4	15	
5	35	
6	25	
TOTAL	**105 (5 pts. bonus)**	

1. (8 points) Evaluate $\lim_{x\to 1^+} x^{1/(1-x)}$

Let $\lim_{x\to 1^+} x^{1/(1-x)} = L$, so $\ln(L) = \lim_{x\to 1^+} \frac{\ln(x)}{1-x}$

$\lim_{x\to 1^+} \frac{\ln(x)}{(1-x)} = \lim_{x\to 1^+} \frac{-1}{x} = -1 \Rightarrow \ln(L) = -1 \Rightarrow L = \frac{1}{e}$

$\left(\frac{0}{0}\right)$ (Use L'Hospital's rule)

Thus $\lim_{x\to 1^+} x^{1/(1-x)} = \frac{1}{e}$ //

2. (7 points) Let $f(x)$ be a differentiable function such that $f(0) = 2$ and $f'(x) \geq 1$ for all $x > 0$. Show that $f(x) \geq x + 2$ for all $x > 0$ (Hint: Use mean value theorem).

By mean value theorem, there exists $c \in (0, x)$ such that

$\frac{f(x) - f(0)}{x - 0} = f'(c) \Rightarrow \frac{f(x) - 2}{x} = f'(c) \geq 1$

$\Rightarrow f(x) - 2 \geq x \Rightarrow f(x) \geq x + 2$ for all $x > 0$.

3. (15 points) Let $F(x) = \int_{\sqrt{x-1}}^{0} \ln(t^2+1)dt$

(a) (8 points) Find the derivative of $F(x)$

By fundamental theorem of calculus,

$$F'(x) = -\ln\left(\left(\sqrt{x-1}\right)^2+1\right)\cdot \frac{d\left(\sqrt{x-1}\right)}{dx} = \frac{-\ln(x)}{2\sqrt{x-1}} \quad //$$

(b) (7 points) Does $F(x)$ have an absolute maximum on the interval $[1,\infty)$? Justify your answer.

For $x \in (1,\infty)$, $F'(x) = \frac{-\ln(x)}{2\sqrt{x-1}} < 0$ $\left(\text{since } \ln(x), \sqrt{x-1} > 0 \text{ on } (1,\infty)\right)$

So $F(x)$ is decreasing on $(1,\infty)$.

x	1	∞
F'		$-$
F		$\searrow$

So the absolute maximum of $F(x)$ is attained at $x=1$;

$$F(1) = \int_0^0 \ln(t^2+1)dt = 0 \quad //$$

4. (15 points) Find all points (x, y) satisfying $y^2 - x^2 = 1$ and $0 \leq x \leq 3$ that are farthest away from the point $(2, 0)$.

The distance from (x,y) to $(2,0)$ is

$$d = \sqrt{(x-2)^2 + (y-0)^2} = \sqrt{(x-2)^2 + x^2 + 1} = \sqrt{2x^2 - 6x + 5}$$

So we need to find the [redacted] <u>absolute</u> <u>maximum</u> of

$f(x) = 2x^2 - 6x + 5$ on $[0, 3]$.

Now $f'(x) = 4x - 6 = 0 \Rightarrow x = 1$ is the only critical number

- $f(1) = 2 - 4 + 5 = 3$
- $f(0) = 5$
- $f(3) = 18 - 12 + 5 = 11$

So the absolute maximum of $f(x)$ on $[0,3]$ is $f(3) = 11$.

Put $x = 3$ in $y^2 = x^2 + 1 \Rightarrow y = -\sqrt{10}$ or $\sqrt{10}$

So the farthest points are $(3, -\sqrt{10})$ and $(3, \sqrt{10})$ //

5. (35 points) Let $f(x) = x^2 + \frac{8}{x}$

(a) (2 points) Find the domain of $f(x)$.

$\mathbb{R} \setminus \{0\}$

(b) (2 points) Find the x and y intercepts of $f(x)$ if they exist.

$x \neq 0$, so $f(x)$ has no y-intercept.

$y = f(x) = 0 \Rightarrow x^2 + \frac{8}{x} = 0 \Rightarrow x^3 + 8 = 0 \Rightarrow x = -2$

So the x-intercept is $(-2, 0)$.

(c) (3 points) Find the horizontal and vertical asymptotes of $f(x)$ if they exist.

- $\lim_{x \to \pm\infty} f(x) = \infty$, so $f(x)$ has no horizontal asymptotes.

- $\lim_{x \to 0^+} \left(x^2 + \frac{8}{x}\right) = +\infty$, so the y-axis is the vertical asymptote.

$\lim_{x \to 0^-} \left(x^2 + \frac{8}{x}\right) = -\infty$

(d) (3 points) Compute $f'(x)$ and $f''(x)$.

$f'(x) = 2x - \frac{8}{x^2} = \frac{2(x^3 - 4)}{x^2}$

$f''(x) = 2 + \frac{16}{x^3} = \frac{2(x^3 + 8)}{x^3}$

(e) (10 points) Find the intervals on which $f(x)$ is increasing or decreasing; and the intervals on which $f(x)$ is concave up or concave down.

$f'(x) = \dfrac{2(x^3-4)}{x^2} = 0 \Rightarrow x = 2^{2/3}$

$f''(x) = \dfrac{2(x^3+8)}{x^3} = 0 \Rightarrow x = -2$

x		-2		0		$2^{2/3}$	
f'	$-$		$-$		$-$		$+$
f''	$+$	0	$-$		$+$		$+$
f	$\searrow \cup$		$\searrow \cap$		$\searrow \cup$		$\nearrow \cup$

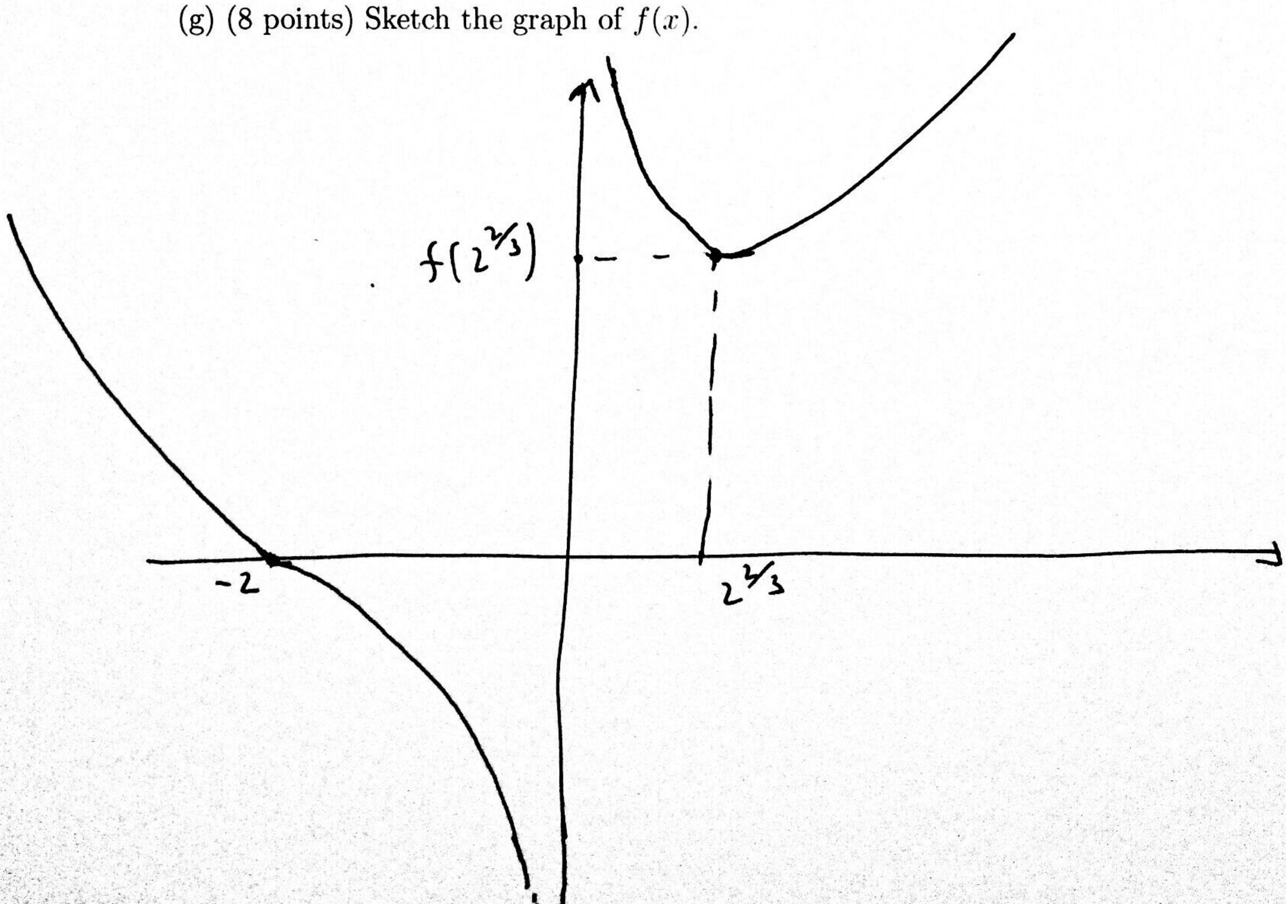
So $f(x)$ is increasing on $(2^{2/3}, \infty)$, and decreasing on $(-\infty, 0), (0, 2^{2/3})$

$f(x)$ is concave up on $(-\infty, -2), (0, \infty)$, and concave down on $(-2, 0)$

(f) (7 points) Determine the local extreme values; and the inflection points of $f(x)$ if they exist.

- $f(x)$ has no local [redacted] maximum.
- $f(x)$ has a local minimum at $x = 2^{2/3}$; $f(2^{2/3}) = 2^{4/3} + 2^{7/3}$
- $(-2, 0)$ is the inflection point.

(g) (8 points) Sketch the graph of $f(x)$.



6. (20 points) Evaluate the following integrals.

(a) (7 points) $\int_{-2}^{-1} \left(x^3 + \frac{1}{x}\right) dx = \left(\frac{x^4}{4} + \ln|x|\right)\Big|_{-2}^{-1}$

$= \frac{1}{4} + \ln(1) - \left(\frac{16}{4} + \ln(2)\right) = \frac{-15}{4} - \ln(2)$ //

(a) (9 points) $\int_{1}^{e} \frac{(\ln x)^2 + \ln x}{x}\, dx = \int_0^1 (u^2+u)\, du$

$\left(u = \ln(x),\ du = \frac{1}{x}\, dx\right)$

$= \left(\frac{u^3}{3} + \frac{u^2}{2}\right)\Big|_0^1 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$ //

(b) (9 points) $\int xe^{-x}\, dx = -x \cdot e^{-x} + \int e^{-x}\, dx$

$\left(\begin{array}{ll} u = x, & dv = e^{-x} dx \\ du = dx, & v = -e^{-x} \end{array}\right)$

$= -xe^{-x} - e^{-x} + C$ //