1. (12 points) Find the area enclosed by the curves $y = x^3$ and $y = 2x^2 - x$. (Simplify your answer.)

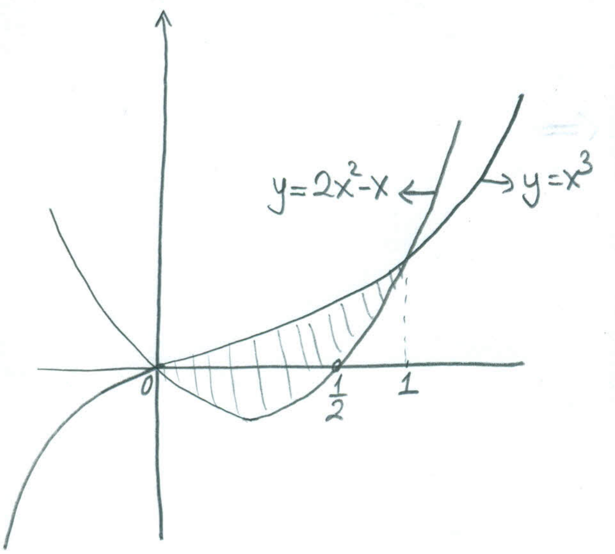
$f(x) = x^3$, $g(x) = 2x^2 - x$

<u>Intersection points:</u> $f(x) - g(x) = 0$

$x^3 - 2x^2 + x = 0$

$x(x^2 - 2x + 1) = x(x-1)^2 = 0$

$\Rightarrow x = 0$ or $x = 1$



x		0		1	
f-g	−		+	0	+

On $[0,1]$, $f-g \geq 0$, that is $f(x) \geq g(x)$

$$\text{Area} = \int_0^1 (f(x) - g(x))\,dx = \int_0^1 (x^3 - 2x^2 + x)\,dx$$

$$= \left.\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2}\right|_0^1$$

$$= \left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right) - 0$$

$$= \frac{1}{12} \mathbin{/\!/}$$

2. Evaluate the following integrals. (Simplify your answers.)

(a) (7 points) $\int 2x\cos(2x+1)\,dx$. Integration by Parts

Let $u=2x$ and $\cos(2x+1)dx = dv$ then $du=2dx$ and $v=\dfrac{\sin(2x+1)}{2}$

$$\int 2x\cos(2x+1)dx = 2x.\frac{\sin(2x+1)}{2} - \int \frac{\sin(2x+1)}{2}.2dx$$

$$= x.\sin(2x+1) - \int \sin(2x+1)\,dx$$

$$= x.\sin(2x+1) + \frac{\cos(2x+1)}{2} + C$$

(b) (7 points) $\int \frac{6-5x}{2x^2+5x-3}\,dx$. Partial Fractions

$$\frac{6-5x}{2x^2+5x-3} = \frac{6-5x}{(2x-1)(x+3)} = \frac{A}{2x-1} + \frac{B}{x+3} \Rightarrow A.(x+3)+B.(2x-1) = 6-5x$$

$$(A+2B)x + (3A-B) = 6-5x$$

$2x \quad -1$

$x \quad 3$

$\Rightarrow A+2B=-5$ and $3A-B=6$

$\Rightarrow A=1, \; B=-3$

$$\int \frac{6-5x}{2x^2+5x-3}dx = \int \frac{1}{2x-1}dx + \int \frac{(-3)}{x+3}dx$$

$$= \frac{\ln|2x-1|}{2} - 3\ln|x+3| + C$$

(c) (8 points) $\int_0^{\pi/2} \sin^3 x \cos^3 x \, dx$. Trigonometric Integral

$\cos^2 x = 1 - \sin^2 x \Rightarrow \cos^3 x = \cos^2 x . \cos x = (1 - \sin^2 x). \cos x$

$$\int_0^{\pi/2} \sin^3 x \cos^3 x \, dx = \int_0^{\pi/2} \sin^3 x . (1 - \sin^2 x) \cos x \, dx = \int_0^1 u^3 (1-u^2) \, du$$

$$= \int_0^1 (u^3 - u^5) \, du$$

$$= \frac{u^4}{4} - \frac{u^6}{6} \Big|_0^1 = \frac{1}{4} - \frac{1}{6} - 0$$

$$= \frac{1}{12}$$

$\left\{ \begin{array}{l} \text{let } u = \sin x \text{, then } du = \cos x \, dx \\ x = 0 \Rightarrow u = \sin 0 = 0, \ x = \frac{\pi}{2} \Rightarrow u = \sin\left(\frac{\pi}{2}\right) = 1 \end{array} \right\}$

(d) (8 points) $\int_0^{\ln 5} \frac{e^x}{2e^x - 1} \, dx$. Substitution

let $u = 2e^x - 1$ then $du = 2e^x \, dx \Rightarrow e^x dx = \frac{1}{2} du$

$x = 0 \Rightarrow u = 1$, $x = \ln 5 \Rightarrow u = 2e^{\ln 5} - 1 = 9$

$$\int_0^{\ln 5} \frac{e^x}{2e^x - 1} \, dx = \frac{1}{2} \int_1^9 \frac{du}{u} = \frac{1}{2} \ln|u| \Big|_1^9 = \frac{1}{2}(\underbrace{\ln 9}_{2\ln 3} - \underbrace{\ln 1}_{0}) = \ln 3$$

3. (a) (6 points) Find the derivative of the function $f(x) = \sqrt{(x^2.5^x)^3}$.

$$f(x) = \left((x^2 . 5^x)^3\right)^{1/2} = \left(x^6 . 5^{3x}\right)^{1/2} = x^3 . 5^{\frac{3x}{2}}$$

$$f'(x) = 3x^2 . 5^{\frac{3x}{2}} + x^3 . 5^{\frac{3x}{2}} . \ln 5 . \frac{3}{2}$$

↑

Product Rule

(b) (6 points) Find dy/dx if $\ln x + \ln(y^2) = 3$. Implicit Differentiation

$$\frac{1}{x} + \frac{1}{y^2} \cdot 2y . y' = 0 \Rightarrow \frac{2y'}{y} = -\frac{1}{x}$$

$$\Rightarrow y' = -\frac{y}{2x} \;//$$

(c) (6 points) Write the equation of the tangent line to the curve $y = 1 - e^x$ at the point where its graph crosses the x-axis.

Point of intersection of $y = 1 - e^x$ with the x-axis $(y=0)$:

$$1 - e^x = 0 \Rightarrow e^x = 1 \Rightarrow x = 0 \Rightarrow P(0,0)$$

<u>Slope:</u> $y' = -e^x$, $y'\big|_{x=0} = -e^0 = -1$

Equation of the tangent line through $(0,0)$ with slope -1 :

$$y - 0 = (-1)(x - 0)$$

$$y = -x \;//$$

4. (11 points) The product of two positive numbers is 54. Find the numbers if the sum of the first number and the square of the second number is as small as possible.

$x \cdot y = 54$, $x > 0$, $y > 0$

$\Downarrow$

$x = \dfrac{54}{y}$

$x + y^2$: minimize

$f(y) = \dfrac{54}{y} + y^2$, $y > 0$

$f'(y) = -\dfrac{54}{y^2} + 2y = \dfrac{2y^3 - 54}{y^2} = \dfrac{2(y^3 - 3^3)}{y^2}$

Only critical point in $(0, \infty)$ is $y = 3$

$f''(y) = \dfrac{108}{y^3} + 2$, $f''(3) = \dfrac{108}{27} + 2 = 6 > 0$

From 2nd derivative test f is minimum at $y=3$

$y = 3 \Rightarrow x = \dfrac{54}{3} = 18$

5. Evaluate the following limits if they exist. If the limit does not exist explain why. State the type of indeterminate form, if any.

(a) (6 points) $\lim_{x\to 2} \frac{|x-2|}{x^2-x-2}$ $\left(\frac{0}{0}\right)$ $x^2-x-2=(x-2)(x+1)$

$$\lim_{x\to 2^+} \frac{|x-2|}{(x-2)(x+1)} = \lim_{x\to 2^+} \frac{x-2}{(x-2)(x+1)} = \lim_{x\to 2^+} \frac{1}{x+1} = \frac{1}{3}$$

$$\lim_{x\to 2^-} \frac{|x-2|}{(x-2)(x+1)} = \lim_{x\to 2^-} \frac{-(x-2)}{(x-2)(x+1)} = \lim_{x\to 2^-} \frac{-1}{x+1} = -\frac{1}{3}$$

Since $\lim_{x\to 2^+} \frac{|x-2|}{x^2-x-2} \neq \lim_{x\to 2^-} \frac{|x-2|}{x^2-x-2}$ limit does not exist!

(b) (6 points) $\lim_{x\to\infty} \sqrt{x^2-x}-x$ $(\infty-\infty)$ Multiply and divide by conjugate:

$$\lim_{x\to\infty} \sqrt{x^2-x}-x = \lim_{x\to\infty} \frac{(\sqrt{x^2-x}-x).(\sqrt{x^2-x}+x)}{(\sqrt{x^2-x}+x)}$$

$$= \lim_{x\to\infty} \frac{(x^2-x)-(x)^2}{\sqrt{x^2-x}+x} = \lim_{x\to\infty} \frac{-x}{|x|\sqrt{1-\frac{1}{x}}+x} = \lim_{x\to\infty} \frac{-x}{x\left(\sqrt{1-\frac{1}{x}}+1\right)}$$

$$= \lim_{x\to\infty} \frac{-1}{\sqrt{1-\frac{1}{x}}+1} = \frac{-1}{\sqrt{1-0}+1} = -\frac{1}{2} //$$

(c) (8 points) $\lim_{x \to 0^+} x^{\sin x}$ (0^0)

Let $y = x^{\sin x}$ then $\ln y = \ln(x^{\sin x}) = \sin x . \ln x$

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \sin x . \ln x \overset{0.(-\infty)}{=} \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{\sin x}} = \lim_{x \to 0^+} \frac{\ln x}{\csc x}$$

$$\underset{l'H.R.}{\overset{\frac{-\infty}{\infty}}{=}} \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\csc x . \cot x} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{\left(-\frac{1}{\sin x}\right) . \frac{\cos x}{\sin x}}$$

$$= \lim_{x \to 0^+} \left(-\frac{\sin x}{x}\right) . \left(\frac{\sin x}{\cos x}\right) = (-1) . \left(\frac{0}{1}\right) = 0$$

$$\lim_{x \to 0^+} y = e^{\lim_{x \to 0^+} \ln y} = e^0 = 1 //$$

6. (a) (6 points) If $F(x) = \int_1^{\sqrt{x}} \sin t^2 dt$ then find $F'(4)$.

Let $u = \sqrt{x}$ then $F(u) = \int_1^u \sin t^2 dt$.

By Fundamental Theorem of Calculus $\frac{dF}{du} = \sin u^2$

By Chain Rule : $\frac{dF}{dx} = \frac{dF}{du} \cdot \frac{du}{dx} = \sin u^2 \cdot \frac{1}{2\sqrt{x}} = \frac{\sin x}{2\sqrt{x}}$

$F'(4) = \frac{dF}{dx}\Big|_{x=4} = \frac{\sin 4}{2\sqrt{4}} = \frac{\sin 4}{4}$ //

(b) (8 points) Evaluate the integral $\int_2^\infty \frac{1}{x(\ln x)^2}\, dx$ (Improper Integral)

$$\int_2^\infty \frac{1}{x(\ln x)^2} dx = \lim_{t \to \infty} \int_2^t \frac{1}{x(\ln x)^2} dx$$

$$\int_2^t \frac{1}{x(\ln x)^2} dx = \int_{\ln 2}^{\ln t} \frac{du}{u^2} = -\frac{1}{u}\Big|_{\ln 2}^{\ln t} = \frac{1}{\ln 2} - \frac{1}{\ln t}$$

$\left\{ \begin{array}{l} \text{let } u = \ln x \text{ then } du = \frac{1}{x} dx \\ x = 2 \Rightarrow u = \ln 2,\ x = t \Rightarrow u = \ln t \end{array} \right\}$

$$\int_2^\infty \frac{1}{x(\ln x)^2} dx = \lim_{t \to \infty} \left(\frac{1}{\ln 2} - \frac{1}{\ln t} \right) = \frac{1}{\ln 2}$$ //