MATH 102 - 2. MIDTERM QUESTIONS & SOLUTIONS

Problem 1 (10 pts) Let $g(x) = \int_{e^x}^{x^3} \cos^3(\ln t)\, dt$. Find $g'(x)$.

Problem 2 Find the area of the region between the following curves.

2a (10 pts) $y = 9 - x^2$ and x-axis.

2b (10 pts) $y = x^3 - 4x$ and $y = 5x$.

Problem 3 (15 pts) Find the following limit: $\lim_{x \to 0^+} x^{\sin x}$

Problem 4 Consider the region between the curves $y = \sqrt{12x}$, x-axis, and $x = 3$.

4a (10 pts) Rotate the region about x-axis. Find the volume of the solid.

4b (15 pts) Rotate the region about the vertical line $x = -2$. Find the volume of the solid.

Problem 5 (15 pts) Find the following integral: $\int \frac{dx}{e^x+1}$

Problem 6 Find the following integrals.

6a (6 pts) $\int e^{2x}. \sin(3x)\, dx$

6b (6 pts) $\int x^2. \sqrt{4x^2 - 9}\, dx$

6c (8 pts) $\int x. \arcsin(2x)\, dx$

MATH 102 - 2. MIDTERM SOLUTIONS:

① Let $G'(t)=\cos^3(\ln t) \Rightarrow g(x)=G(x^3)-G(e^x) \Rightarrow g'(x)=G'(x^3).3x^2-G'(e^x).e^x$

$\Rightarrow \underline{g'(x)=\cos^3(\ln x^3).3x^2-\cos^3(\ln e^x).e^x}$

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② 2a $9-x^2=0 \Rightarrow x=-3,\ x=3 \Rightarrow \text{Area}=\int_{-3}^{3}9-x^2\,dx=9x-\frac{x^3}{3}\Big|_{-3}^{3}=(27-9)-(-27+9)$
$=18+18=\underline{36}$

2b $x^3-4x=5x \Rightarrow x^3-9x=0 \Rightarrow x.(x-3)(x+3)=0 \Rightarrow x=-3,0,3$

$\text{Area}=\int_{-3}^{0}(x^3-4x)-5x\,dx+\int_{0}^{3}5x-(x^3-4x)\,dx=\int_{-3}^{0}x^3-9x\,dx+\int_{0}^{3}9x-x^3\,dx$

$=\frac{x^4}{4}-\frac{9x^2}{2}\Big|_{-3}^{0}+\frac{9x^2}{2}-\frac{x^4}{4}\Big|_{0}^{3}=\frac{81}{4}+\frac{81}{4}=\frac{81}{2}$ //

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③ $y=x^{\sin x} \Rightarrow \ln y=\ln x^{\sin x}=\sin x.\ln x \Rightarrow \lim_{x\to0^+}\ln y=\lim_{x\to0^+}\sin x.\ln x=\lim_{x\to0^+}\frac{\ln x}{\frac{1}{\sin x}}$

$\underset{\text{L'hospital}}{=}\lim_{x\to0^+}\frac{\frac{1}{x}}{\frac{-\cos x}{\sin^2 x}}=\lim_{x\to0^+}\frac{-\sin^2 x}{x.\cos x}\underset{\text{L'hopital}}{=}\lim_{x\to0^+}\frac{2\sin x.\cos x}{\cos x-x\sin x}=\frac{0}{1}=0$ $\quad \ln y\to0$

$\Rightarrow y\to1$ //

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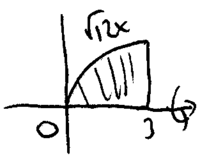
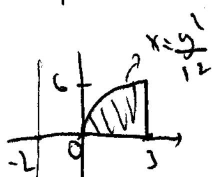
④ 4a



$V=\int_{0}^{3}\pi(\sqrt{12x})\,dx=12\pi\int_{0}^{3}x\,dx=6\pi x^2\Big|_{0}^{3}=54\pi$ //

4b



$V=\pi\int_{0}^{6}\left(5^2-\left(\frac{y^2}{12}+2\right)^2\right)dy=\pi\int_{0}^{6}25-\left(\frac{y^4}{144}+\frac{y^2}{3}+4\right)dy=\pi\int_{0}^{6}\frac{-y^4}{144}-\frac{y^2}{3}+21\,dy$

$y=\sqrt{12x}\Rightarrow x=\frac{y^2}{12}$

$=\pi\left[\frac{-y^5}{720}-\frac{y^3}{9}+21y\right]\Big|_{0}^{6}=\pi\left[\frac{-54}{5}-24+126\right]=\frac{456\pi}{5}$ //

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⑤ $\int\frac{dx}{e^x+1}=\int\frac{1}{u+1}.\frac{du}{u}=\int\frac{du}{u(u+1)}=\int\frac{1}{u}-\frac{1}{u+1}\,du=\ln u-\ln(u+1)=\ln e^x-\ln(e^x+1)+C$

$=\underline{x-\ln(e^x+1)+C}$

$u=e^x$

$du=e^x dx \Rightarrow dx=\frac{du}{u}$

$\frac{1}{u.(u+1)}=\frac{1}{u}-\frac{1}{u+1}$

(6) (6a) $\int e^{2x} \cdot \sin 3x \, dx = \dfrac{e^{2x}}{13}(2\sin 3x - 3\cos 3x) + C$

$\boxed{107}$ $a=2$ $b=3$

(6b) $\int x^2 \cdot \sqrt{4x^2-9}\, dx = 2\int x^2 \cdot \sqrt{x^2-\frac{9}{4}}\, dx = 2\left[\frac{x}{8}\left(2x^2-\frac{9}{4}\right)\cdot\sqrt{x^2-\frac{9}{4}} + \frac{81}{64}\ln\left|x+\sqrt{x^2-\frac{9}{4}}\right|\right]+C$

$\boxed{41}$ $a=\frac{3}{2}$

(6c) $\int x \cdot \arcsin 2x \, dx = \dfrac{x^2}{2}\arcsin 2x - \dfrac{2}{2}\int \dfrac{x^2 dx}{\sqrt{1-4x^2}}\, dx$

$\boxed{99}$ $n=1$ $a=2$

$= \dfrac{x^2}{2}\arcsin 2x - \dfrac{1}{2}\int \dfrac{x^2 dx}{\sqrt{\frac{1}{4}-x^2}}$

$\boxed{33}$ $a=\frac{1}{2}$

$= \dfrac{x^2}{2}\arcsin 2x - \dfrac{1}{2}\left[\dfrac{1}{8}\arcsin 2x - \dfrac{1}{2}x\sqrt{\frac{1}{4}-x^2}\right] + C$