FALL 2006 MIDTERM 2 SOLUTIONS

1) a) By Fundamental Theorem of Calculus

$$\frac{d}{dx}\left(\int_x^{x^2-1} \sin^3(7t-1)\,dt\right) = \sin^3(7(x^2-1)-1) - \sin^3(7x-1)$$
$$= \sin^3(7x^2-8) - \sin^3(7x-1)$$

b) $$\int_0^1 e^x(x^2-3) = e^x(x^2-2x-1)\Big|_0^1$$
$$= e^1(1^2-2.1-1) - e^0(0^2-2.0-1)$$
$$= -2e+1$$

2) a) Let $x=u$, $\cos x\,dx = dv$

$\Rightarrow du = 1$ and $v = \sin x$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$$
$$= x\sin x - (-\cos x + c) \quad \text{for some } c \in \mathbb{Z}$$
$$= x\sin x + \cos x + C$$

b) $$\frac{4x^2}{(x+1)^2(x^2+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{(x+1)^2} + \frac{E}{x+1}$$
$$= \frac{(Ax+B)(x+1)^2 + C(x^2+1) + E(x^2+1)(x+1)}{(x^2+1)(x+1)^2}$$
$$= \frac{(A+E)x^3 + (2A+B+C+E)x^2 + (A+2B+E)x + (B+C+E)}{(x^2+1)(x+1)^2}$$

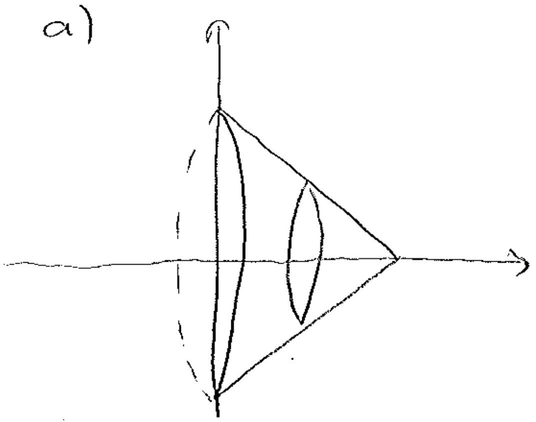
$$\Rightarrow \begin{array}{l} A+E=0 \\ 2A+B+C+E=4 \\ A+2B+E=0 \\ B+C+E=0 \end{array} \quad \Rightarrow \quad \begin{array}{l} A=2 \\ B=0 \\ C=2 \\ E=-2 \end{array}$$

$$\Rightarrow \int \frac{4x^2}{(x^2+1)(x+1)^2}\,dx = \int \frac{2x}{x^2+1}\,dx + \int \frac{2}{(x+1)^2}\,dx + \int \frac{-2}{x+1}\,dx$$

$$= \left(\ln|x^2+1| + C_1\right) + \left(-2(x+1)^{-1} + C_2\right) + \left(-2\ln|x+1| + C_3\right)$$

$$= \ln|x^2+1| - 2(x+1)^{-1} - 2\ln|x+1| + C$$

3) a)



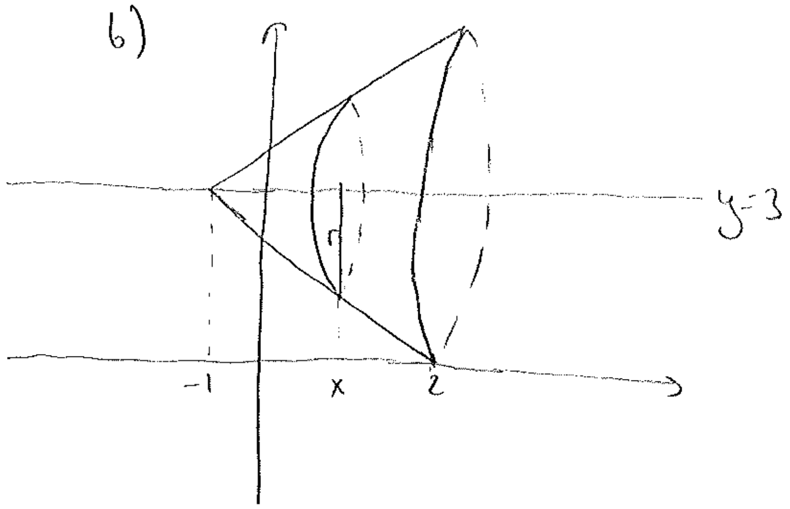
The area of the cross section is:

$A = \pi(-x+2)^2$

So the volume is:

$$\int_0^2 \pi(-x+2)^2\,dx$$

b)



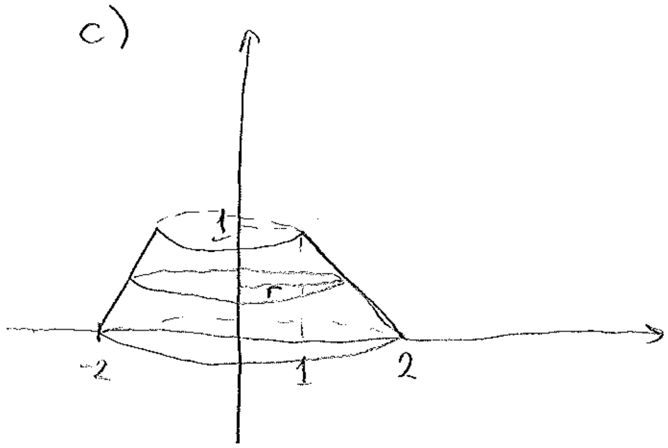
$r = 3 - (-x+2)$

So $A = \pi\,(3-(-x+2))^2$

Hence the volume is:

$$\int_{-1}^{2} \pi\,(1+x)^2\,dx$$

c)

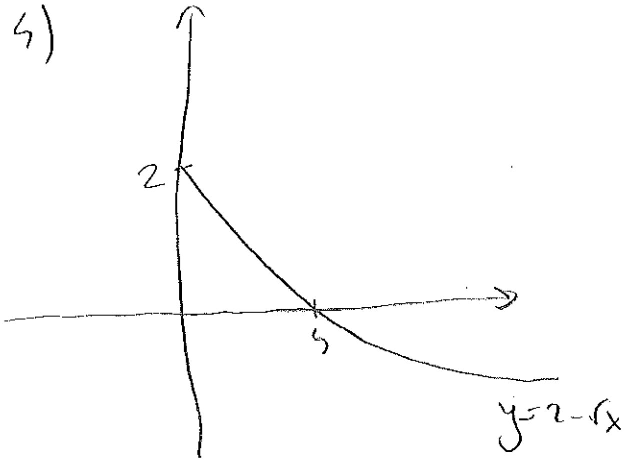


$y = -r+2 \Rightarrow r = 2-y$

$A = \pi\,(2-y)^2$

$$\int_0^1 \pi\,(2-y)^2\,dy$$

4)



$$A = \int_0^4 (2-\sqrt{x})\,dx$$

$$= \left(2x - \frac{2}{3}\cdot x^{3/2} + C\right)\Big|_0^4$$

$$= 8 - \frac{16}{3} = \frac{8}{3}$$

5) a) $f(x) = 2^x$

$\ln f(x) = \ln 2^x = x \cdot \ln 2$

$\frac{\ln f(x)}{\ln 2} = x$

$f^{-1}(x) = \frac{\ln x}{\ln 2}$

$\frac{d f^{-1}(x)}{dx} = \frac{d}{dx}\left(\frac{\ln x}{\ln 2}\right) = \frac{1}{x} \cdot \frac{1}{\ln 2}$

b) $f(x) = \log_3(x)$

$3^{f(x)} = x$

$f^{-1}(x) = 3^x$

$\frac{d f^{-1}(x)}{dx} = \frac{d 3^x}{dx} = 3^x \cdot \ln 3$

c) $f(x) = 2\sin x$

$\frac{1}{2} f(x) = \sin x$

$\arcsin\left(\frac{1}{2} f(x)\right) = x$

$f^{-1}(x) = \arcsin \frac{1}{2} x$

$\frac{d f^{-1}(x)}{dx} = \frac{d(\arcsin \frac{1}{2} x)}{dx}$

$= \frac{1}{2} \cdot \frac{1}{\sqrt{1 - \frac{1}{4}x^2}}$

6) a) $f(x) = x^{\pi}$

$\int x^{\pi} dx = \frac{1}{\pi+1} x^{\pi+1} + C$

b) $f(x) = \pi^x$

$\int \pi^x dx = \frac{1}{\ln \pi} \cdot \pi^x + C$

c) $f(x) = 6\tan 3x$

$\int 6 \tan 3x \, dx = \int 6 \frac{\sin 3x}{\cos 3x} dx$

$\cos 3x = a \qquad -3\sin 3x\, dx = da$

$\Rightarrow \int \frac{6\sin 3x}{\cos 3x} dx = \int -2\frac{1}{a} \cdot da$

6) c) $f(x) = 6\tan 3x$

$$\int 6\tan 3x\,dx = \int 6\,\frac{\sin 3x}{\cos x}\,dx$$

Let $\cos 3x = a$

$\Rightarrow -3\sin 3x\,dx = da$

$$\Rightarrow \int 6\tan 3x\,dx = \int -2\cdot\frac{1}{a}\cdot da = -2\ln|a| + C$$

$$= -2\ln|\cos 3x| + C$$

7) a) $\int \cos^3 x \cdot \sin x \cdot dx$ $\quad \cos x = a \Rightarrow -\sin x\,dx = da$

$$\Rightarrow \int -a^3\,da = -\frac{1}{4}a^4 + C = -\frac{1}{4}\cos^4 x + C$$

b) $$\int \frac{dx}{x^2-2x+2} = \int \frac{dx}{(x^2-2x+1)+1} = \int \frac{1}{(x-1)^2+1}\,dx$$

$$= \arctan(x-1) + C$$

c) $$\int \frac{2x^3}{x^2-1}\,dx = \int\left(\frac{2x}{x^2-1} + 2x\right)dx = \int \frac{2x}{x^2-1}\,dx + \int 2x\,dx$$

$x^2 - 1 = a$

$2x\,dx = da$

So $$\int \frac{2x^3}{x^2-1}\,dx = \int \frac{1}{a}\cdot da + \int 2x\,dx$$

$$= \ln|a| + C_1 + x^2 + C_2$$

$$= \ln|x^2-1| + x^2 + C$$

d) $$\int \frac{\log_2(x^2)}{x}\,dx = \int \frac{\ln(x^2)}{\ln 2}\cdot\frac{1}{x}\,dx$$

$\left.\begin{array}{l} \ln x^2 = a \\ \frac{2}{x}dx = da \end{array}\right\} \Rightarrow$

$$= \int \frac{1}{\ln 2}\frac{1}{2}\cdot a\cdot da = \frac{1}{2\ln 2}\int a\,da$$

$$= \frac{1}{2\ln 2}\cdot\frac{1}{2}a^2 + C = \frac{1}{4\ln 2}\cdot(\ln x^2)^2 + C$$