KOÇ UNIVERSITY

MATH 102 - CALCULUS

Midterm II December 10, 2009

Duration of Exam: 90 minutes

INSTRUCTIONS: No calculators may be used on the test. No books, no notes, and no talking allowed. You must always **explain your answers** and **show your work** to receive **full credit**. Use the back of these pages if necessary. **Print (use CAPITAL LETTERS) and sign your name, and indicate your section below.**

Surname, Name: ANSWERS

Signature: ________________

Section (Check One):

Section 1: S. Ünver (MW, 9:30) ___
Section 2: S. Ünver (MW, 15:30) ___
Section 3: Y. Arkun (TT, 12:30) ___

PROBLEM	POINTS	SCORE
1	10	
2	5	
3	30	
4	15	
5	15	
6	10	
7	15	
TOTAL	**100**	

Problem 1. (10 pts) Compute the derivatives of the following functions:

(a) (5 pts) $\sqrt{2sin^2(3x)+5}$

(b) (5 pts) $ln(sin(e^x)))$

We use Chain Rule.

a) $$\frac{d}{dx}\sqrt{2\sin^2(3x)+5} = \frac{1}{2\sqrt{2\sin^2(3x)+5}} \cdot \frac{d}{dx}(2\sin^2(3x)+5)$$

$$= \frac{1}{2\sqrt{2\sin^2(3x)+5}} \cdot \left(2 \cdot 2\sin(3x) \cdot \frac{d}{dx}(\sin(3x)) + 0\right)$$

$$= \frac{2\sin(3x)\cos(3x) \cdot \frac{d}{dx}(3x)}{\sqrt{2\sin^2(3x)+5}}$$

$$= \frac{3\sin(6x)}{\sqrt{2\sin^2(3x)+5}}$$

b) $$\frac{d}{dx}(\ln(\sin(e^x))) = \frac{1}{\sin(e^x)} \cdot \frac{d}{dx}(\sin(e^x)) = \frac{1}{\sin(e^x)} \cdot \cos(e^x) \cdot \frac{d}{dx}(e^x)$$

$$= \frac{e^x\cos(e^x)}{\sin(e^x)} = e^x \cdot \tan(e^x)$$

Problem 2. (5 pts) Find the following limit:

$$\lim_{x\to 0} \frac{1-\cos(x)}{\sin(x)} \underset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{\sin(x)}{\cos(x)} = 0 \quad \text{since } \sin(x) \to 0 \text{ and } \cos(x) \to 1 \text{ as } x \to 0.$$

L' Hospital Rule.

Problem 3. (30 pts) Let $f(x) = x^3 - 6x + 2$.

(a) (5 pts) Find the intervals on which f is increasing/decreasing.

(b) (5 pts) Find the local maximum/minimum point(s) of f.

(c) (5 pts) Find the intervals on which f is concave up/down.

(d) (5 pts) Find the inflection point(s) of f.

(e) (10 pts) Sketch a graph of f.

a) $f'(x) = 3x^2 - 6 = 0 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$.

		$-\sqrt{2}$		$\sqrt{2}$	
f'	+	0	−	0	+
f	↗		↘		↗

f is incresing on $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, +\infty)$

f is decresing on $(-\sqrt{2}, \sqrt{2})$.

b) local maximumpoint: $(-\sqrt{2},\ 4\sqrt{2}+2)$

local minimum point: $(\sqrt{2},\ -4\sqrt{2}+2)$

c) $f''(x) = 6x$

		0	
f''	−	0	+
f	concave down		concave up

f is concave up on $(0, +\infty)$

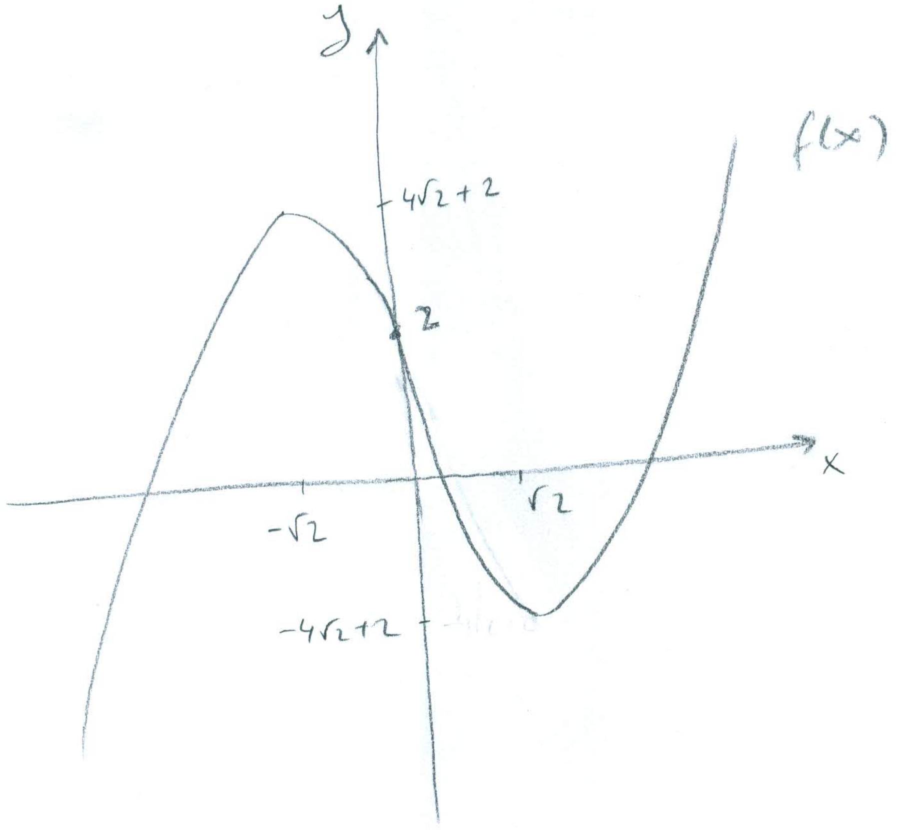
f is concave down on $(-\infty, 0)$

d) inflection point $(0, 2)$.

e) $\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = -\infty$ So no horizontal asymptote.

Domain: $\mathbb{R}$ no vertical asymptote.



Problem 4. (15 pts.) The following cost function is given

$$C(x) = 16000 + 200x + 4x^{3/2},$$

where x represents the production level (units produced). Find the production level that will minimize the average cost and find the corresponding minimum average cost.

$C(x)$: cost function

$p(x)$: demand (or price) function

$R(x) = x\,p(x)$: Revenue function

$P(x) = R(x) - C(x)$: Profit function.

$AC(x) = $ Average cost function $= \dfrac{C(x)}{x} = \dfrac{16000}{x} + 200 + 4x^{1/2}$ → minimize this

$$\frac{d\,AC(x)}{dx} = -\frac{16000}{x^2} + 2x^{-1/2}$$

If $\frac{d}{dx}AC(x) = 0$ then $x^{-1/2} = \dfrac{8000}{x^2} \Rightarrow x^{3/2} = 8000 \Rightarrow x = (8000)^{2/3} = 400$

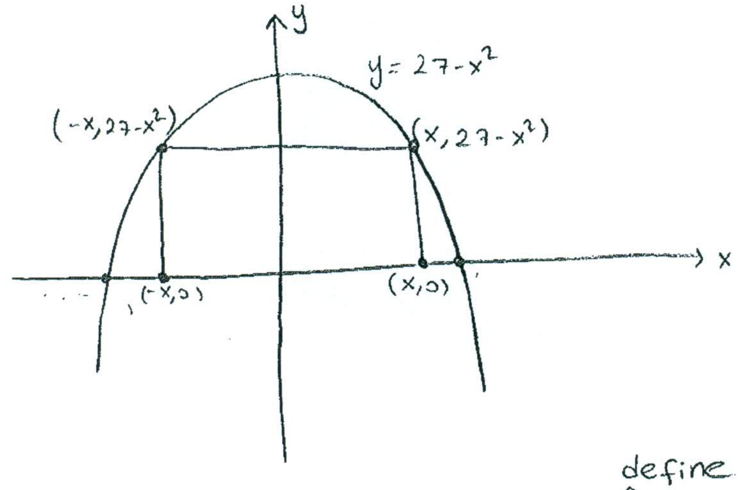
$$\frac{d^2 AC(x)}{dx} = \frac{32000}{x^3} - x^{-3/2}$$

$$\frac{d^2 AC(400)}{dx} = \frac{32000}{(400)^3} - (400)^{-3/2} = \frac{\cancel{32000}}{64.\cancel{000000}} - (20)^{-3} = \frac{1}{2000} - \frac{1}{8000} = \frac{4000-1}{8000} > 0$$

By 2nd Derivative test, 400 is local minimum of $AC(x)$.

So $AC(400) = \dfrac{1600\cancel{0}}{40\cancel{0}} + 200 + 4\sqrt{400} = 40 + 200 + 80 = 320$ is the corresponding minimum average cost.

Problem 5. (15 pts.) Find the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis and lying on the parabola $y = 27 - x^2$.



Area of the rectangle $= x \cdot (27 - x^2) = -x^3 + 27x \overset{\text{define}}{=:} f(x)$

We need to maximize $f(x)$.

$f'(x) = -3x^2 + 27$

If $f'(x) = 0$ then $3x^2 = 27 \Rightarrow x^2 = 9 \Rightarrow |x| = \pm 3$

$f''(x) = -6x$

$f(3) = -18 < 0$ $\longrightarrow$ So $(3, 27 - 3^2) = (3, 18)$ maximizes $f(x)$

$f(-3) = 18 > 0$

which means that dimensions of the rectangle of largest area is 3 and 18.

Problem 6. (10 pts.) Find $\frac{dy}{dx}$ by implicit differentiation if

$$x cos(y) + y cos(x) = 1.$$

We differentiate both sides of the equation:

$$\frac{d}{dx}\left(x\cos(y) + y\cos(x)\right) = \frac{d}{dx}(1)$$

$$\overbrace{1.\cos(y) + x(-\sin(y)).\frac{dy}{dx}}^{\text{by product rule}} + \overbrace{\frac{dy}{dx}\cos(x) + y(-\sin(x))}^{\text{by product rule}} = 0$$

$$\Rightarrow \cos y - x\sin(y)\frac{dy}{dx} + \frac{dy}{dx}\cos(x) - y\sin(x) = 0$$

$$\Rightarrow \frac{dy}{dx}\left(\cos(x) - x\sin(y)\right) = y\sin(x) - \cos(y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y\sin(x) - \cos(y)}{\cos(x) - x\sin(y)}$$

Problem 7. (15 pts) Evaluate the following integrals:

(a) (5 pts) $\int_0^1 \sqrt{x}dx$

(b) (5 pts) $\int_{-1}^1 |x^3|dx$

(c) (5 pts) $\int_0^{ln(2)} e^x dx$

a) $\int_0^1 \sqrt{x}\, dx = \int_0^1 x^{1/2}\, dx = \left[\frac{2}{3} x^{3/2}\right]_0^1 = \frac{2}{3} 1^{3/2} - \frac{2}{3} 0^{3/2} = \frac{2}{3}$

b) $\int_{-1}^1 |x^3|\, dx = \begin{cases} \int_{-1}^1 x^3\, dx & \text{if } x \geq 0 \\ -\int_{-1}^1 x^3\, dx & \text{if } x < 0 \end{cases}$

$= \begin{cases} \frac{x^4}{4} & \text{if } x \geq 0 \\ \frac{-x^4}{4} & \text{if } x < 0 \end{cases}$

c) $\int_0^{\ln(2)} e^x\, dx = \left[e^x\right]_0^{\ln 2} = e^{\ln 2} - e^0 = 2 - 1 = 1$