MATH 102 FINAL EXAM SPRING 2008

Problem 1 (10 pts) Find the following limit.

$$\lim_{n\to\infty} \frac{\ln(n+1)}{\sqrt{n}}$$

Problem 2 (10 pts) Find the equation of the tangent line of the curve at $(0, 2)$.

$$y = \frac{\ln(\cos x) + 2}{e^x}$$

Problem 3 (15 pts) Let $f(x) = x.e^x$
a) Find all critical points, and intervals on which f is increasing & decreasing.
b) Find inflection points, and intervals on which f is concave up & concave down.
c) Find the asymptotes, if exist.
d) Sketch the graph of f.

Problem 4 (15 pts) Compute the following improper integral.

$$\int_1^\infty x.e^{-x}\ dx$$

Problem 5 (10 pts) Find two numbers such that their difference is 18 and their product is minimum.

Problem 6 (15 pts) Compute the following integral.

$$\int_0^1 \ln(x^2+1)\ dx$$

Problem 7 (10 pts) Find the area of the region between the curves $y = \sqrt{8x}$ and $y = x^2$.

Problem 8 (15 pts) Find the volume of the solid obtained by rotating only one region between $y = \sqrt{2\sin(2x)}$ and $y = 0$ about x-axis.

MATH 102 FINAL SOLUTIONS:

① $\lim\limits_{n\to\infty}\dfrac{\ln(n+1)}{\sqrt{n}} \underset{\text{L'Hospital}}{=} \lim\limits_{n\to\infty}\dfrac{\frac{1}{n+1}}{\frac{1}{2\sqrt{n}}} = \lim\limits_{n\to\infty}\dfrac{2\sqrt{n}}{n+1} = \lim\limits_{n\to\infty}\dfrac{2\sqrt{n}}{n(1+\cancel{\frac{1}{n}})} = \lim\limits_{n\to\infty}\dfrac{2\sqrt{\cancel{n}}}{\cancel{n}} = \lim\limits_{n\to\infty}\dfrac{2}{\sqrt{n}} = 0$

② $y = \dfrac{\ln(\cos x)+2}{e^x}$ $\quad y' = \dfrac{\frac{\sin x}{\cos x}\cdot e^x - (\ln\cos x + 2)\cdot e^x}{e^{2x}}$ $\quad x=0 \Rightarrow m = \dfrac{0\cdot 1 - 2\cdot 1}{1} = -2$

$\dfrac{y-2}{x-0} = -2$ $\quad y = -2x+2$

③ a. $f'(x) = e^x + x\cdot e^x = (x+1)e^x \Rightarrow x=-1$

f': $-$ | $+$ at -1 → local min.

$f'>0$ $(-1,\infty)$ ↗

$f'<0$ $(-\infty,-1)$ ↘

b. $f''(x) = e^x + (x+1)e^x = (x+2)e^x$

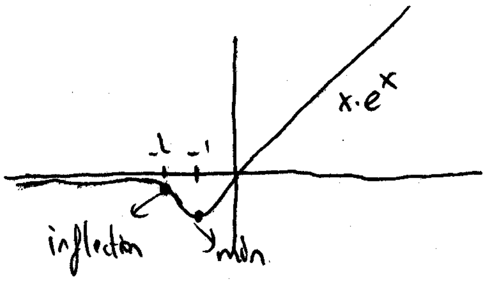
f'': $-$ | $+$ at -2 → inflection pt.

$f''>0$ $(-2,\infty)$ concave up

$f''<0$ $(-\infty,-2)$ concave down

c. $\lim\limits_{x\to\infty} x\cdot e^x = \infty$ $\quad \lim\limits_{x\to-\infty} x\cdot e^x = \lim\limits_{x\to-\infty}\dfrac{x}{e^{-x}} \underset{\text{L'Hospital}}{=} \lim\limits_{x\to-\infty}\dfrac{1}{-e^{-x}} = 0$ $\quad y=0$ hor. asymptote.

d.



④ $\displaystyle\int_1^\infty x\cdot e^{-x}\,dx = \lim_{t\to\infty}\int_1^t x\cdot e^{-x}\,dx = \lim_{t\to\infty}\left(-(x+1)e^{-x}\right)\Big|_1^t = \lim_{t\to\infty} -(t+1)\cdot e^{-t} + \frac{2}{e} = \lim_{t\to\infty} -\frac{(t+1)}{e^t} + \frac{2}{e}$

$\displaystyle \underset{\text{L'Hospital}}{=} \lim_{t\to\infty}\frac{1}{e^t} + \frac{2}{e} = 0 + \frac{2}{e} = \frac{2}{e}$

$\displaystyle\int x\cdot e^{-x}\,dx = -x e^{-x} + \int e^{-x}\,dx = -(x+1)e^{-x}$

(105) $n=1$, $a=-1$ or (104) $a=-1$

⑤ $x-y=18 \Rightarrow y = x-18$

$f(x) = x\cdot y = x\cdot(x-18) = x^2-18x \Rightarrow f'(x) = 2x-18 \Rightarrow x=9$, $y=-9$

⑥ $\int_0^1 \ln(x^2+1)\,dx = x\cdot\ln(x^2+1) - \int \frac{2x^2}{x^2+1}\,dx = x\cdot\ln(x^2+1) - 2(x-\arctan x)\Big|_0^1 = \big(\ln 2 - 2(1-\underbrace{\arctan 1}_{\pi/4})\big) - (0-0)$

$u = \ln(x^2+1) \quad dv = dx$

$du = \frac{2x}{x^2+1}\,dx \quad v = x$

$\int \frac{x^2}{x^2+1}\,dx = \int 1 - \frac{1}{x^2+1}\,dx = x - \arctan x$

$= \ln 2 - 2 + \frac{\pi}{2}$ //

⑦ $y = \sqrt{8x}$

$y = x^2$

$x^2 = \sqrt{8x}$

$x^4 = 8x$

$x^4 - 8x = 0$

$x(x^3-8) = 0 \Rightarrow x = 0,\ x = 2$

$A = \int_0^2 \sqrt{8x} - x^2\,dx = \sqrt{8}\cdot\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\Big|_0^2$

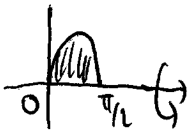
$= \frac{4\sqrt{2}}{3}\sqrt{x^3} - \frac{x^3}{3}\Big|_0^2 = \frac{16}{3} - \frac{8}{3} = \frac{8}{3}$ //

⑧ $y = \sqrt{2\sin 2x}$

$y = 0$

$\sqrt{2\sin 2x} = 0$

$\Rightarrow x = 0, \pm\frac{\pi}{2}, \pm\pi, \ldots, \pm k\frac{\pi}{2}$



$R = \sqrt{2\sin 2x}$

One region $0 \le x \le \frac{\pi}{2}$

$V = \pi\int_0^{\pi/2} \left(\sqrt{2\sin 2x}\right)^2 dx = \pi\int_0^{\pi/2} 2\sin 2x\,dx$

$= \pi(-\cos 2x)\Big|_0^{\pi/2}$

$= \pi\left[(+1) - (-1)\right] = 2\pi$ //