Final Spring 2006

Problem 1. Calculate the following limits or show that they do not exist.

a) $\lim_{x\to 1} \frac{(x-1)\sqrt{2x}}{|3x-3|} = \lim_{x\to 1} \frac{(x-1)\sqrt{2x}}{|x-1|\ 3}$

$\lim_{x\to 1^+} \frac{x-1}{|x-1|} \frac{\sqrt{2x}}{3} = \lim_{x\to 1^+} \frac{x-1}{x-1} \frac{\sqrt{2x}}{3} = \frac{2}{3}$ Right limit is not equal to left limit

$\lim_{x\to 1^-} \frac{x-1}{|x-1|} \frac{\sqrt{2x}}{3} = \lim_{x\to 1^-} \frac{x-1}{1-x} \frac{\sqrt{2x}}{3} = -\frac{2}{3}$ # which means then the limit does not exist.

b) $\lim_{x\to 0} \frac{\ln(1+x)}{\cos x} = \frac{\ln 1}{\cos 0} = \frac{0}{1} = 0$

c) For which values of a, the function

$$f(x) = \begin{cases} x^2 - 2x & \text{for } x \le a \\ x^2 & \text{for } x > a \end{cases}$$

is continuous at $x = a$?

$f(x)$ is continuous at $x = a \Leftrightarrow \lim_{x\to a} f(x) = f(a)$

$\Rightarrow \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = \lim_{x\to a} f(x) = f(a) = a^3 - 2a$

$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} x^2 = a^2 \Rightarrow a^2 = a^3 - 2a \Rightarrow a^3 - a^2 - 2a = 0$
$\Rightarrow a(a^2 - a - 2) = 0$

$\Rightarrow a = 0$ or $a^2 - a - 2 = 0$
$\Delta = 1 + 8 = 9 \Rightarrow a_1 = \frac{1+\sqrt{9}}{2} = \frac{4}{2} = 2$, $a_2 = \frac{1-\sqrt{9}}{2} = -\frac{2}{2} = -1$

So for $a=0, a=2$ and $a=-1$, $f(x)$ is continuous at a.

Problem 2) Find the derivative of the following function f in (a)-(c). Simplify your answers.

a) (8pts) $f(x) = e^{x^2+2} + \ln\left(\frac{x+1}{x^2+2}\right)$

$$f'(x) = e^{x^2+2} \cdot (2x) + \frac{1}{\frac{x+1}{x^2+2}} \cdot \left(\frac{x+1}{x^2+2}\right)'$$

$$= 2xe^{x^2+2} + \frac{x^2+2}{x+1} \cdot \frac{x^2+2-(x+1)2x}{(x^2+2)^2}$$

$$= 2xe^{x^2+2} + \frac{x^2+2-2x^2-2x}{(x+1)(x^2+2)}$$

$$= 2xe^{x^2+2} + \frac{-x^2-2x+2}{(x+1)(x^2+2)}$$

Problem 2)

b) (6pts) $f(x) = (\sin x)^{\cos x}$

$$y = (\sin x)^{\cos x}$$

$$\ln y = \cos x \ \ln(\sin x)$$

$$\frac{y'}{y} = -\sin x \ \ln(\sin x) + \cos x \cdot \frac{1}{\sin x} \cdot \cos x$$

$$\frac{y'}{y} = -\sin x \ln(\sin x) + \frac{\cos^2 x}{\sin x}$$

So $$y' = \left(-\sin x \ln(\sin x) + \frac{\cos^2 x}{\sin x}\right)(\sin x)^{\cos x}$$

Problem 2)

c) (6 pts) $f(x)=\sqrt[3]{\dfrac{1+x^3}{1-x^3}}=\left(\dfrac{1+x^3}{1-x^3}\right)^{1/3}$

$$f'(x)=\frac{1}{3}\left(\frac{1+x^3}{1-x^3}\right)^{\frac{1}{3}-1}\cdot\left(\frac{1+x^3}{1-x^3}\right)'$$

$$=\frac{1}{3}\left(\frac{1+x^3}{1-x^3}\right)^{-1/3}\cdot\frac{3x^2(1-x^3)+(1+x^3)(3x^2)}{(1-x^3)^2}$$

$$=\frac{1}{3}\,\frac{(1-x^3)^{2/3}}{(1+x^3)^{2/3}}\;\frac{3x^2(1-x^3+1+x^3)}{(1-x^3)^2}$$

$$=\frac{(1-x^3)^{\frac{2}{3}-2}\,2x^2}{(1+x^3)^{2/3}}=\frac{(1-x^3)^{-4/3}\cdot 2x^2}{(1+x^3)^{2/3}}=\frac{2x^2}{\sqrt[3]{(1+x^3)^2\,(1-x^3)^4}}$$

d) (5 pts) Find the equation of the tangent line at the point $P(1,e)$ to the curve defined by the equation $y=e^{1/x}$.

$y=f(x)=e^{1/x}$

$y'=f'(x)=e^{1/x}\cdot\left(\frac{1}{x}\right)'=-e^{1/x}\,\frac{1}{x^2}=-\frac{e^{1/x}}{x^2}$

$f'(1)=-\frac{e^1}{1}=-e \rightsquigarrow$ slope of the tangent line

$\Rightarrow$ Equation of the tangent line at $P(1,e)$ is

$y-e=-e(x-1)$

$y=-ex+2e$

Problem 3, Consider the function

$$f(x) = \frac{e^x}{x+2}$$

a) (6 pts) Find the horizontal and vertical asymptotes of the graph of f if they exist.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{e^x}{x+2} \underset{\frac{\infty}{\infty}}{\overset{\text{L'Hospital}}{=}} \lim_{x\to\infty} e^x = \infty$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{e^x}{x+2} \underset{\frac{\infty}{\infty}}{=} \lim_{x\to-\infty} e^x = \infty$$

There is no horizontal asymptote.

$f(x) = \frac{e^x}{x+2}$ is undefined if $x+2=0 \iff x=-2$, So

$$\lim_{x\to-2^+} f(x) = \lim_{x\to-2^+} \frac{e^x}{x+2} = \infty$$

$$\lim_{x\to-2^-} f(x) = \lim_{x\to-2^-} \frac{e^x}{x+2} = -\infty$$

So $x=-2$ is the vertical asymptotes

b) (2 pts) Find the intervals on which the function f is increasing and decreasing.

$$f'(x) = \frac{e^x(x+2) - e^x}{(x+2)^2} = \frac{e^x(x+1)}{(x+2)^2}$$

f is increasing means $f'(x) > 0 \Rightarrow \frac{e^x(x+1)}{(x+2)^2} > 0 \Rightarrow x+1>0 \iff x>-1$

So f is increasing on $(-1,\infty)$.

f is decreasing means $f'(x) < 0 \Rightarrow \frac{e^x(x+1)}{(x+2)^2} < 0 \iff x+1<0 \iff x<-1$

So f is decreasing on $(-\infty,-1)$

c) (2 pts) Determine the local extreme values of the function f.

f has a local maximum if $f'(x)=0$ and $f''(x)<0$

f has a local minimum if $f'(x)=0$ and $f''(x)>0$.

So $f'(x)=\dfrac{e^x(x+1)}{(x+2)^2}$

$$f''(x)=\frac{(e^x(x+1)+e^x)(x+2)^2-2(x+2)e^x(x+1)}{(x+2)^2}$$

$$=\frac{e^x(x+2)^2-2e^x(x+1)}{(x+2)}$$

$$=\frac{e^x[(x+2)^2-2(x+1)]}{x+2}$$

$$=\frac{e^x(x^2+2x+4-2x-2)}{x+2}=\frac{e^x(x^2+2)}{x+2}$$

$f'(x)=\dfrac{e^x(x+1)}{(x+2)^2}=0 \Rightarrow x+1=0 \Rightarrow \underbrace{x=-1}$ critical point.

And $f'(-2)$ does not exists, but since f is undefined at $x=-2$,

it is not a critical point.

$f''(-1)=\dfrac{e^{-1}((-1)^2+2)}{-1+2}=\dfrac{3}{e}>0$ so f has a local minimum at $x=-1$.

So $f(-1)=\dfrac{e^{-1}}{-1+2}=\dfrac{1}{e}$ is the local minimum values of the

function f

Problem 4) Calculate the following integrals.

a) (5 pts) $\int \frac{1}{x^2} \sin\frac{1}{x}\, dx$

$$u = \frac{1}{x} \qquad du = -\frac{1}{x^2}\, dx$$

$$= -\int \sin u \, du$$

$$= -(-\cos u) + C = \cos\frac{1}{x} + C$$ for some constant C.

b) (8pts) $\int \frac{3x^2+4x+4}{x^3+x}\, dx = \int \frac{3x^2+4x+4}{x(x^2+1)}\, dx = \int \left(\frac{A}{x} + \frac{Bx+C}{x^2+1}\right) dx$

$$\Rightarrow \frac{A(x^2+1)+x(Bx+C)}{(x^2+1)\,x} = \frac{3x^2+4x+4}{x(x^2+1)}$$

$$\Rightarrow Ax^2 + A + Bx^2 + Cx = 3x^2+4x+4$$

$$\Rightarrow (A+B)x^2 + Cx + A = 3x^2+4x+4$$

$$\Rightarrow \left.\begin{array}{l} A+B = 3 \\ C = 4 \\ A = 4 \end{array}\right\} B = -1$$

$$u = x^2+1 \qquad du = 2x\,dx$$

$$\Rightarrow \int \frac{3x^2+4x+4}{x^3+x}\, dx = \int \left(\frac{4}{x} + \frac{-x+4}{x^2+1}\right) dx = \int \left(\frac{4}{x} - \frac{x}{x^2+1} + \frac{4}{x^2+1}\right) dx$$

$$= 4\ln|x| - \frac{1}{2}\int \frac{1}{u}\, du + 4\arctan|x| + C_1 = 4\ln|x| - \frac{1}{2}\ln|x^2+1| + 4\arctan|x| + C_2$$

for some constants C_1 and C_2.

Problem 4)

c) (7 pts) $\int e^x \sin x \, dx$ $\quad \begin{pmatrix} u = e^x & v = -\cos x \\ du = e^x dx & dv = \sin x \, dx \end{pmatrix}$

$= -e^x \cos x + \int e^x \cos x \, dx$ $\quad \begin{pmatrix} u = e^x & v = \sin x \\ du = e^x dx & dv = \cos x \, dx \end{pmatrix}$

$= -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$

$\Rightarrow 2\int e^x \sin x \, dx = e^x(\sin x - \cos x)$

$\Rightarrow \int e^x \sin x \, dy = \dfrac{e^x(\sin x - \cos x)}{2}$