KOÇ UNIVERSITY

MATH 106 - CALCULUS I

Midterm II December 9, 2013

Duration of Exam: 75minutes

INSTRUCTIONS: CALCULATORS ARE NOT ALLOWED FOR THIS EXAM. No books, no notes, no questions and no talking allowed. You must always **explain your answers** and **show your work** to receive **full credit**. Use the back of these pages if necessary. **Print (use CAPITAL LETTERS) and sign your name, and indicate your section below.**

Surname, Name: ______________________

Signature: KEY

Section (Check One):

Lecture 1 (Haluk Oral	–	MW 14:00-15:15)	:	—
Lecture 2 (Selda Küçükçifçi	–	MW 11:00-12:15)	:	—
Lecture 3 (Selda Küçükçifçi	–	MW 12:30-13:45)	:	—
Lecture 4 (Haluk Oral	–	MW 15:30-16:45)	:	—

PROBLEM	POINTS	SCORE
1	18	
2	25	
3	20	
4	23	
5	14	
BONUS	8	
TOTAL	**108**	

1. (a) (10 points) Compute the limit

$$\lim_{x\to 1^+} x^{1/(1-x)} =$$

$$y = x^{1/1-x}$$

$$\ln y = \frac{1}{1-x} \ln x$$

$$\lim_{x\to 1^+} \ln y = \lim_{x\to 1^+} \frac{\ln x}{1-x} \underset{\frac{0}{0}}{=} \lim_{x\to 1^+} \frac{\frac{1}{x}}{-1} = -1.$$

$$\text{So} \quad \lim_{x\to 1^+} y = e^{-1}$$

b) (8 points) If f' is continuous, $f(4) = 0$ and $f'(4) = 2$, evaluate

$$\lim_{x\to 0} \frac{f(4+3x) + f(4+4x)}{x} \underset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{f'(4+3x) \cdot 3 + f'(4+4x) \cdot 4}{1}$$

$$= 3f'(4) + 4f'(4) = 7 \cdot 2 = 14$$

2. (25 points) Find the critical points, the intervals where function is increasing and decreasing, local minimum, maximum and inflection points, if they exist and the intervals where the function is concave up and down for the function $f(x) = x + 2\cos x$ on the interval $[0, 2\pi]$.

$f(x) = x + 2\cos x \qquad 0 \le x \le 2\pi$

$f'(x) = 1 - 2\sin x = 0 \Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \quad x = \frac{5\pi}{6}.$

$f''(x) = -2\cos x = 0 \Rightarrow \quad \cos x = 0 \Rightarrow x = \frac{\pi}{2}, \quad x = \frac{3\pi}{2}.$

	0		$\pi/6$		$\pi/2$		$5\pi/6$		$3\pi/2$		2π
f'		$+$	0	$-$		$-$	0	$+$		$+$	
f''		$-$		$-$	0	$+$		$+$	0	$-$	
f		↗		↘		↘		↗		↗	

f is increasing on $(0, \frac{\pi}{6}) \cup (\frac{5\pi}{6}, 2\pi)$

f is decreasing on $(\frac{\pi}{6}, \frac{5\pi}{6})$

f is concave up on $(\frac{\pi}{2}, \frac{3\pi}{2})$

f is concave down on $(0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$

local maximum point : $(\frac{\pi}{6}, \frac{\pi}{6} + \sqrt{3})$

local minimum point : $(\frac{5\pi}{6}, \frac{5\pi}{6} - \sqrt{3})$

inflection points : $(\frac{\pi}{2}, \frac{\pi}{2})$ & $(\frac{3\pi}{2}, \frac{3\pi}{2})$

3. (20 points) Consider two non-negative numbers x and y. Suppose $x + 2y = 100$. Find x and y

(a) such that their product is maximum,

(b) such that their product is minimum.

(a) $P = xy = (100-2y)y$

$P(y) = 100y - 2y^2$

$P'(y) = 100 - 4y = 0 \Rightarrow y = 25$

Since $P(y)$ is a quadratic function such that

its vertex $(50, 25)$ maximizes $P(y)$,

$x = 25$
$y = 50$ maximizes the product.

(b) $P(y) = 100y - 2y^2$ $\quad 0 \leq y \leq$

$y = 0$ $\quad x = 100$

& $y = 50$ $\quad x = 0$ $\quad$ minimize the product

Since the lowest product of two non-negative numbers can be 0.

4. (a) (12 points) Find a function f such that $f''(x) = x + \cos x$, $f(0) = 1$ and $f'(0) = 2$.

$$f'(x) = \frac{x^2}{2} + \sin x + C$$

$$f'(0) = 2 = 0 + C \Rightarrow C = 2$$

$$f(x) = \frac{x^3}{6} - \cos x + 2x + D$$

$$f(0) = 1 = -1 + D \Rightarrow D = 2$$

So $$f(x) = \frac{x^3}{6} - \cos x + 2x + 2 .$$

(b) (11 points) Evaluate $\dfrac{d}{dx}\displaystyle\int_{\cos x}^{\sin x} \frac{1-t^2}{1+t^4}\,dt$

$$\frac{d}{dx}\int_{\cos x}^{\sin x} \frac{1-t^2}{1+t^4}\,dt = \frac{d}{dx}\left(-\int_0^{\cos x} \frac{1-t^2}{1+t^4}\,dt + \int_0^{\sin x} \frac{1-t^2}{1+t^2}\,dt\right)$$

$$= -\frac{1-\cos^2 x}{1+\cos^4 x}\cdot(-\sin x) + \frac{1-\sin^2 x}{1+\sin^4 x}\cdot\cos x$$

$$= \frac{\sin^3 x}{1+\cos^4 x} + \frac{\cos^3 x}{1+\sin^4 x} .$$

5. (14 points) Suppose that f is continuous on $[0, 4]$, $f(0) = 1$, and $2 \leq f'(x) \leq 5$ for all x in $(0, 4)$. Show that $9 \leq f(4) \leq 21$.

By Mean value theorem there is $c \in (0,4)$

such that $f'(c) = \dfrac{f(4) - f(0)}{4 - 0} = \dfrac{f(4) - 1}{4}$

Since $2 \leq f'(x) \leq 5$ then $2 \leq f'(c) \leq 5$ too.

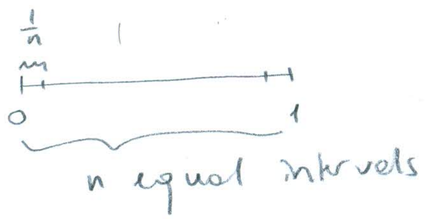
Then $2 \leq \dfrac{f(4) - 1}{4} \leq 5$

$8 \leq f(4) - 1 \leq 20$

$9 \leq f(4) \leq 21$.

BONUS (8 points) Express the following limit as a definite integral on the interval $[0, 1]$ and evaluate it.

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{i^4}{n^5} = \int_0^1 x^4 \, dx = \left[\frac{x^5}{5} \right]_0^1 = \frac{1}{5}.$$



$$\sum_{i=1}^{n} \frac{i^4}{n^5} = \underbrace{\left(\frac{i}{n}\right)^4}_{f(x_i)} \cdot \underbrace{\frac{1}{n}}_{\Delta x}$$