KOÇ UNIVERSITY

MATH 106 - CALCULUS I

Midterm I October 22, 2014

Duration of Exam: 75 minutes

INSTRUCTIONS: CALCULATORS ARE NOT ALLOWED FOR THIS EXAM. No books, no notes, no questions and no talking allowed. You must always **explain your answers** and **show your work** to receive **full credit**. Use the back of these pages if necessary. **Print (use CAPITAL LETTERS) and sign your name, and indicate your section below.**

Surname, Name: ______________________________

Signature: ______________________________

Section (Check One):

Section 1: E. Ş. Yazıcı (Mon-Wed 16:00) ——
Section 2: E. Ş. Yazıcı (Mon-Wed 13:00) ——
Section 3: Doğan Bilge (Mon-Wed 11:30) ——
Section 4: Doğan Bilge (Mon-Wed 14:30) ——
Section 5: Altan Erdoğan (Tu-Th 16:00) ——

PROBLEM	POINTS	SCORE
1	40	
2	16	
3	15	
4	15	
5	15	
TOTAL	**101**	

1. Compute the following limits if they exist. You are not allowed to use the L'Hospital Rule. Specify any infinite limits.

a) (8 points) $\lim_{x\to 0} \frac{\sqrt{\cos x + x^2} - \sqrt{\cos x - x^2}}{sin^2 x} =$

$$\lim_{x\to 0} \frac{\cos x + x^2 - \cos x + x^2}{(\sin^2 x)\left(\sqrt{\cos x + x^2} + \sqrt{\cos x - x^2}\right)}$$

$$= \lim_{x\to 0} \frac{x^2}{\sin^2 x} \cdot \left(\frac{2}{\sqrt{\cos x + x^2} + \sqrt{\cos x - x^2}}\right)$$

$$= \lim_{x\to 0} \left(\frac{2}{\sqrt{\cos x + x^2} + \sqrt{\cos x - x^2}}\right) \Big/ \left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2 = 1.$$

b) (8 points) $\lim_{x\to 0} \frac{1}{|x-1| - |x+1|} =$ $\lim_{x\to 0} \frac{1}{(1-x) - (x+1)} = \lim_{x\to 0} \frac{-1}{2x}$

since $x-1 < 0$ and $x+1 > 0$ if $x \to 0$.

$$\lim_{x\to 0^+} \frac{-1}{2x} = -\infty$$

$$\lim_{x\to 0^-} \frac{-1}{2x} = +\infty$$

c) (8 points) $\lim_{x\to\infty} \frac{x - x^3}{x^2 + 5} =$ $\lim_{x\to\infty} \frac{\frac{1}{x} - x}{1 + 5/x^2} = -\infty.$

d) (8 points) $\lim_{x\to 1} \dfrac{x+1}{1-x^4} =$ $x+1>0, 1-x^4<0$ if $x\to 1^+$

$x+1>0, 1-x^4>0$ if $x\to 1^-$ so

$$\lim_{x\to 1^+} \frac{x+1}{1-x^4} = -\infty$$

$$\lim_{x\to 1^-} \frac{x+1}{1-x^4} = +\infty$$

e) (8 points) $\lim_{x\to 1} \dfrac{\ln x}{x-1} =$ $\lim_{x\to 1} \dfrac{\ln x - \ln 1}{x-1} = (\ln')(1)$

$$= \frac{1}{1} = 1$$

2. Differentiate the following functions. (Note: $\sin^{-1} x = \arcsin x$)

a) (8 points) $f(x) = \ln \dfrac{\sin(2x)}{\sin^{-1} x}$ $= \ln(\sin(2x)) - \ln(\sin^{-1} x)$

$$f'(x) = \frac{2\cos(2x)}{\sin(2x)} - \frac{1}{(\sin^{-1} x)\sqrt{1-x^2}}.$$

b) (8 points) $f(x) = \cos(2^{x^2})$ $f'(x) = -\sin(2^{x^2}) \cdot \frac{d}{dx}(2^{x^2})$

$= -\sin(2^{x^2})(2x \ln 2 \cdot 2^{x^2})$

$= -2^{x^2+1} \ln(2) \, x \, \sin(2^{x^2})$

3. (15 points) Let $f(x) = e^x$ and $g(x) = \sin(\ln x)$. Find equations for two distinct parallel lines l_1 and l_2 where; l_1 is tangent to f at $x = 0$ and l_2 is tangent to g.

$f'(0) = e^0 = 1$ so $\ell_1 : y = 1 \cdot (x-0) + f(0)$

$= x + 1$

$g'(x) = \frac{\cos(\ln x)}{x}$.

Since ℓ_1 is parallel to ℓ_2, they have the same slope. so need to find a st $g'(a) = 1$

ie $\frac{\cos(\ln a)}{a} = 1$ we can choose $a = 1$.

$\ell_2 : y = 1 \cdot (x-a) + g(1) = x - 1$.

4. (15 points) Show that there exist a sphere with radius $r \in (0, 1)$ and a cube with side $r + \frac{1}{2}$ with the same volume. (Volume of the sphere with radius r is $\frac{4}{3}\pi r^3$, and the volume of a cube with side a is a^3).

Volume of sphere of radius r is

$$V_S(r) = \frac{4}{3}\pi r^3.$$

Volume of cube of side $r+\frac{1}{2}$ is

$$V_C(r) = \left(r+\frac{1}{2}\right)^3.$$

Let $f(r) = V_S(r) - V_C(r)$. Then f is continuous everywhere, particularly on $[0,1]$.

if $r = 0$ then $V_C(r) = \frac{1}{8}$, $V_S(r) = 0$

hence $f(0) < 0$.

if $r = 1$ then $V_C(r) = \frac{27}{8} < \frac{32}{8} = 4$

$$V_S(r) = \frac{4\pi}{3} > \frac{4 \cdot 3}{3} = 4$$

so $f(1) > 0$. By IVT, $\exists\, r \in (0,1)$ st $f(r) = 0$
ie $V_C(r) = V_S(r)$.

5. (15 points) Let $x > 0$. Find y' using implicit differentiation if $x^y + y - 1 = 0$.

$$\frac{d}{dx}\left(x^y + y - 1\right) = 0 \implies$$

$$\frac{d}{dx}\left(e^{y\ln x}\right) + \frac{dy}{dx} = 0 \implies$$

$$x^{y-1}\left(x\ln x \cdot \frac{dy}{dx} + y\right) + \frac{dy}{dx} = 0$$

$$\implies \left(x^y \ln x + 1\right)\frac{dy}{dx} = -y x^{y-1}$$

$$\implies \frac{dy}{dx} = \frac{-y x^{y-1}}{x^y \ln x + 1}.$$