Problem 1. Calculate the following limits (specify infinite limits):

(a) (5 pts) $\lim_{x \to \pi^-} \dfrac{x^2 - 8x + 1}{\sin x} = -\infty$

As $x \to \pi^-$ $\sin x \to 0^+$

and $\lim_{x \to \pi^-} (x^2 - 8 + 1) < 0$; hence the above result.

(b) (5 pts) $\lim_{x \to \infty} \sqrt{x^2 + 1} - x$

$$= \lim_{x \to \infty} \frac{\left(\sqrt{x^2+1} - x\right)\left(\sqrt{x^2+1} + x\right)}{\left(\sqrt{x^2+1} + x\right)} =$$

$$= \lim_{x \to \infty} \frac{x^2 + 1 - x^2}{\sqrt{x^2+1} + x} = \frac{1}{\infty} = 0$$

(c) (5 pts) $\lim_{x \to 0} \ln(\tan^2 x) = -\infty$

As $x \to 0$ $\tan^2 x \to 0^+$

and as $\tan^2 x \to 0^+$ $\ln(\tan^2 x) \to -\infty$

Hence the above result.

(d) (5 pts) $\lim_{x \to 1} \dfrac{x^3 - x}{x^2 - 1}$

$$= \lim_{x \to 1} \frac{x\cancel{(x^2-1)}}{\cancel{x^2-1}} = \underline{1}$$

Problem 2. **a.** (10 pts) Let

$$f(x) = \begin{cases} x^2 + 3x + 6 & \text{for} \quad x \leqslant 1 \\ 2x + c^2 & \text{for} \quad x > 1 \end{cases}$$

where c is a fixed real number.

Find all the values of c such that f is continuous for all real numbers.

Since x^2+3x+6 and $2x+c^2$ are polynomials they're continuous on $(-\infty,1)$ and $(1,\infty)$, respectively.

For continuity at $x=1$ we need $f(1) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$.

$f(1) = 1+3+6 = 10$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2+3x+6 = 10$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2x+c^2 = 2+c^2$

$\Rightarrow 2+c^2 = 10 \Rightarrow c^2 = 8$

$c = \pm 2\sqrt{2}$

b. (15 pts) Let $f(x) = x^3 - x^2 + x$. Show that there exists a real number c such that $f(c) = 10$.

Since f is a polynomial its continuous everywhere.

$f(2) = 8-4+2 = 6 < 10$

$f(3) = 27-9+3 = 21 > 10$

Hence by Intermediate Value Theorem

there exist $c \in (2,3)$ s.t. $f(c) = 10$.

Problem 3. Find the derivative of the following function f.

(a) (5 pts) $f(x) = \sin(x^2)$

$$f'(x) = 2x\cos(x^2)$$

(b) (5 pts) $f(x) = \dfrac{1}{\sqrt{1+\tan x}}$

$$f'(x) = -\frac{1}{2\cos^2 x} \cdot \frac{1}{(1+\tan x)^{3/2}} = -\frac{\sec^2 x}{2\sqrt{(1+\tan x)^3}}$$

(c) (5 pts) $f(x) = \dfrac{\cos x}{2+\sin x}$

$$f'(x) = \frac{-\sin x\,(2+\sin x) - \cos^2 x}{(2+\sin x)^2} = \frac{-2\sin x \underbrace{-\sin^2 x - \cos^2 x}_{=-1}}{(2+\sin x)^2}$$

$$\Rightarrow f'(x) = -\frac{1+2\sin x}{(2+\sin x)^2}$$

(d) (5 pts) $f(x) = 1 + x^2 e^{-x}$

$$f'(x) = 0 + 2x\,e^{-x} + x^2 . (-e^{-x})$$
$$= e^{-x}(2x - x^2)$$

Problem 4. (15 pts) Let f and g be two differentiable functions. Assume $r = fog$ and;
$g'(1) = 3$, $g(1) = 5$ and $f'(5) = 11$. Compute

$$\lim_{h \to 0} \frac{r(h+1) - r(1)}{h} = r'(1)$$

$r(t) = f(g(t))$

$r'(t) = g'(t) \cdot f'(g(t))$

$\Rightarrow r'(1) = g'(1) \cdot f'(g(1)) = g'(1) \cdot f'(5) = 3 \cdot 11 = 33$

Problem 5. (20 pts) Use implicit differentiation to find the equation of the tangent line at the point $(3, 1)$ to the curve defined by the equation

$$2(x^2 + y^2)^2 = 25(x^2 - y^2)$$

$$\frac{d}{dx}\left(2(x^2+y^2)^2\right) = \frac{d}{dx}\left(25(x^2-y^2)\right)$$

$$\Rightarrow 2.2 \cdot (x^2+y^2) \cdot (2x+2yy') = 25(2x-2yy') \;;\; \text{Put in } \begin{matrix} x=3 \\ y=1 \end{matrix}$$

$$\Rightarrow 4 \cdot \underbrace{\cancel{(9+1)}}_{2}(6+2y'(3)) = \underbrace{\cancel{25}}_{5}(6-2y'(3))$$

$$\Rightarrow 48 + 16y'(3) = 30 - 10y'(3)$$

$$\Rightarrow 26y'(3) = -18$$

$$\Rightarrow y'(3) = \frac{-18}{26} = \frac{-9}{13}$$

$$\Rightarrow f(x) = -\frac{9}{13}x + b \quad ; f(3) = 1$$

$$\Rightarrow t(3) = -\frac{27}{13} + b = 1 \quad \Rightarrow b = \frac{40}{13}.$$

$$\Rightarrow t(x) = -\frac{9}{13}x + \frac{40}{13}$$