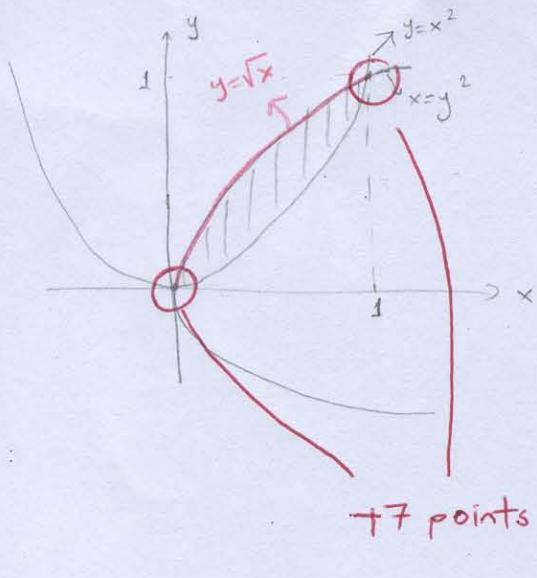


Problem 1 (15 points)

Find the area between the curves $y = x^2$ and $x = y^2$.



$$y = x^2 = y^4 \Rightarrow y^4 - y = 0 \Rightarrow y(y^3 - 1) = 0 \Rightarrow y=0 \text{ or } y=1$$

Therefore the points of intersection of the given curves are $(0,0)$ and $(1,1)$. If we denote the area between the curves as A , then $A = \int_0^1 (\sqrt{x} - x^2) dx = \int_0^1 \sqrt{x} dx - \int_0^1 x^2 dx = \frac{2}{3}x^{3/2} \Big|_0^1 - \frac{1}{3}x^3 \Big|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

+5 points

Problem 2 (15 points)

Calculate the following integrals. Show all your work.

(a) (7 points) $\int_1^e \frac{\sin(\ln x)}{x} dx.$

$$\ln x = u \Rightarrow \frac{1}{x} dx = du. \text{ Then } \int_1^e \frac{\sin(\ln x)}{x} dx = \int_0^1 \sin u du = -\cos u \Big|_0^1 = -\cos 1 + 1 = 1 - \cos 1.$$

$\ln x = u \Rightarrow \frac{1}{x} dx = du$

Then

$\int_0^1 \sin u du = -\cos u \Big|_0^1 = -\cos 1 + 1$

$= 1 - \cos 1$

+4 points

+3 points

(b) (8 points) Evaluate $\int_1^4 e^{\sqrt{x}} dx.$

$$\sqrt{x} = u \Rightarrow \frac{1}{2\sqrt{x}} dx = du \Rightarrow dx = 2\sqrt{x} du = 2u du. \text{ Then } \int_1^4 e^{\sqrt{x}} dx = \int_1^4 e^u \cdot 2u du. \text{ This last integral may be handled via the technique of integration by parts as follows:}$$

+1 points

+1 points

+2 points

$$\left. \begin{array}{l} t = 2u \quad dv = e^u du \\ dt = 2 du \quad v = e^u \end{array} \right\} \int_1^2 2u e^u du = 2u e^u \Big|_1^2 - \int_1^2 2e^u du = 4e^2 - 2e - 2e^u \Big|_1^2 = 4e^2 - 2e - 2e^2 + 2e = 2e^2.$$

+3 points

+1 points

We conclude that $\int_1^4 e^{\sqrt{x}} dx = 2e^2.$

Problem 3 (10 points)

Use calculus to find two non-negative numbers such that their sum is 1 and sum of their squares is as small as possible.

$x, y \geq 0$ such that $x+y=1$. We want to minimize the function f defined as $f(x,y) = x^2 + y^2$. Due to the condition $x+y=1$, we can express f as a function of x only as $f(x) = x^2 + (1-x)^2 = 2x^2 - 2x + 1$, which is valid for $x \geq 0$.

(2 points)

(2 points)

Note that we're searching for the absolute minimum of f on the closed interval $[0,1]$.

Since f is continuous on that interval, we need to check only the critical points and the end points of the given interval. f is a polynomial hence differentiable on $[0,1]$.

Differentiating we find $f'(x) = 4x - 2$. As a result, the critical point of f is (x_0, y_0) where $f'(x_0) = 0$. Solving for x_0 we find $4x_0 - 2 = 0 \Rightarrow x_0 = \frac{1}{2}$ hence $y_0 = 2x_0^2 - 2x_0 + 1 \Rightarrow y_0 = \frac{1}{2}$. The critical point of f is $(\frac{1}{2}, \frac{1}{2})$.

(3 points)

2 points Checking the end points we get $(0,1)$ and $(1,0)$. Among these three points, $(\frac{1}{2}, \frac{1}{2})$ is the one with the smallest value for f hence we conclude that the absolute minimum value of f is $\frac{1}{2}$, taken at the point $x = \frac{1}{2}$.

As a conclusion, $x = y = \frac{1}{2}$ are those numbers that we're looking for.

Problem 4 (15 points)

Use calculus to prove that the line $y = 4x + 7$ intersects the curve $y = 3 \sin x$ at exactly one point.

Let us define a function $f: \mathbb{R} \rightarrow \mathbb{R}$ as $f(x) = 3 \sin x - 4x - 7$. Our aim is to prove that this function has exactly one root. 5 points

First of all, let us show that f has at least one root. f is continuous on \mathbb{R} hence on any interval, the hypotheses of the intermediate value theorem hold. For $x = -\frac{3\pi}{2}$, $f(-\frac{3\pi}{2}) = 3 \sin(-\frac{3\pi}{2}) - 4(-\frac{3\pi}{2}) - 7 = 3 + 6\pi - 7 = 6\pi - 4 > 0$. for $x = \frac{\pi}{2}$, $f(\frac{\pi}{2}) = 3 \sin(\frac{\pi}{2}) - 4\frac{\pi}{2} - 7 = 3 - 2\pi - 7 = -2\pi - 4 < 0$. As a result of the intermediate value theorem, we conclude that there exists $c \in (-\frac{3\pi}{2}, \frac{\pi}{2})$ with $f(c) = 0$.

Then we should show that f has at most one root. Assume that f has two distinct roots x_1 and x_2 with $x_1 < x_2$. f is differentiable on \mathbb{R} hence it satisfies all the necessary hypotheses for Rolle's theorem (or the mean value theorem) on $[x_1, x_2]$. By Rolle's theorem, then, we conclude that there exists $x_0 \in (x_1, x_2)$ with $f'(x_0) = 0$. By differentiating f and putting x_0 in place of x we conclude that

$$f'(x) = 3 \cos x - 4 \Rightarrow f'(x_0) = 3 \cos x_0 - 4 \text{ and } f'(x_0) = 0 \Rightarrow 3 \cos x_0 - 4 = 0$$
$$\Rightarrow \cos x_0 = \frac{4}{3} > 1 \text{ which is impossible}$$

Thus our assumption that "f has two distinct roots" should be false. This observation finishes our work.

Problem 5 (20 points)

(a) (13 points) Determine the radius and the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(5x-3)^n}{n}$.

$a_n = \frac{(5x-3)^n}{n}$ $\Rightarrow \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(5x-3)^{n+1}}{n+1} \right| = \frac{n}{n+1} |5x-3|$ which converges to $|5x-3|$ as $n \rightarrow \infty$.
 By the ratio test, $\sum a_n$ converges absolutely for $|5x-3| < 1$ and diverges for $|5x-3| > 1$.

Rearranging $|5x-3| < 1$ as $|x - \frac{3}{5}| < \frac{1}{5}$ we find the radius of convergence as $R = \frac{1}{5}$. We're sure about the convergence of the series $\sum_{n=1}^{\infty} \frac{(5x-3)^n}{n}$ on $(\frac{2}{5}, \frac{4}{5})$. In order to determine the interval of convergence, we need to check the behavior of the numerical series resulting from putting $x = \frac{2}{5}$ and $x = \frac{4}{5}$ in the given series. For $x = \frac{2}{5}$ we obtain the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is convergent by the alternating series test. For $x = \frac{4}{5}$ we obtain

the famous harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ which is divergent by the p-test. As a result the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(5x-3)^n}{n}$ is $[\frac{2}{5}, \frac{4}{5}]$.

+2 points

(b) (7 points) Represent the function $f(x) = \frac{1}{3x+4}$ as a power series.

$$f(x) = \frac{1}{3x+4} = \frac{1}{4+3x} = \frac{1}{4} \cdot \frac{1}{1+\frac{3x}{4}} = \frac{1}{4} \cdot \frac{1}{1-(-\frac{3x}{4})} = \frac{1}{4} \cdot \sum_{n=0}^{\infty} \left(-\frac{3x}{4}\right)^n \text{ for } \left|-\frac{3x}{4}\right| < 1$$

$$\text{or } f(x) = \sum_{n=0}^{\infty} \frac{(-3)^n x^n}{4^{n+1}} \text{ valid for } |x| < \frac{4}{3}$$

Any attempt to use $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \rightarrow +5 \text{ points}$

Correct Taylor series formula for a general f
 $\rightsquigarrow +4 \text{ points}$

Problem 6 (20 points)

Suppose that the series $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -5$ and diverges when $x = 10$. For each of the following series, determine whether it is convergent or divergent; you need to justify your answer for full credit. (If a definite answer cannot be determined from the given information, give examples to explain why.)

(a) (4 points) $\sum_{n=0}^{\infty} c_n$

$\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$ where R is the radius of convergence. Since $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -5$, $R \geq 5$. Hence $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = 1$. Thus $\sum_{n=0}^{\infty} c_n$ converges.

(b) (4 points) $\sum_{n=0}^{\infty} (-1)^n c_n 11^n$

Since $\sum_{n=0}^{\infty} c_n x^n$ diverges when $x = 10$, $R \leq 10$.

Hence for $x = -11$, $|x| > R$ and therefore

$$\sum_{n=0}^{\infty} (-1)^n c_n 11^n \text{ diverges.}$$

(c) (4 points) $\sum_{n=0}^{\infty} c_n(-4)^n$

Since $R \geq 5$, $| -4 | = 4 < R$. Hence
 $\sum_{n=0}^{\infty} c_n (-4)^n$ converges.

(d) (8 points) $\sum_{n=0}^{\infty} c_n 7^n$

We have $10 \geq R \geq 5$. From this information we can not decide whether $\sum_{n=0}^{\infty} c_n 7^n$ converges or not, since $R > 7$ or $R < 7$ both may happen.

Problem 7 (15 points)

Determine whether the series below are convergent or divergent. Justify your answer by explicitly stating what test you are appealing to and how you use that test.

(a) (5 points) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ Since $\frac{1}{n(\ln n)^2} > 0$ for every $n \geq 2$

and $f(x) = \frac{1}{x(\ln x)^2}$ is continuous decreasing for all $x \geq N$ we can apply integral test.

$$\text{Hence } \int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \lim_{b \rightarrow \infty} \left(-\frac{1}{\ln x} \right) \Big|_2^b = \frac{1}{\ln 2}$$

Thus $\int_2^{\infty} \frac{1}{x(\ln x)^2} dx$ converges, so $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.

(b) (5 points) $\sum_{n=1}^{\infty} \frac{1}{1 + \sin(1/n)}$

Since $\lim_{n \rightarrow \infty} \frac{1}{1 + \sin(1/n)} = \frac{1}{1 + \lim_{n \rightarrow \infty} \sin(1/n)} = 1 \neq 0$, by n -th term

test $\sum_{n=1}^{\infty} \frac{1}{1 + \sin(1/n)}$ diverges.

(c) (5 points) $\sum_{n=2}^{\infty} \frac{n^2}{n^4 - 1}$ Since $\frac{n^2}{n^4 - 1} < \frac{n^2 + 1}{n^4 - 1}$, we have

$$\sum_{n=2}^{\infty} \frac{n^2}{n^4 - 1} < \sum_{n=2}^{\infty} \frac{n^2 + 1}{n^4 - 1} = \sum_{n=2}^{\infty} \frac{1}{n^2 - 1}. \text{ Now } \frac{1}{n^2 - 1} = \frac{1}{(n-1)(n+1)} = \frac{1}{2} \left(\frac{1}{n-1} - \frac{1}{n+1} \right)$$

Hence $\sum_{n=2}^{\infty} \frac{1}{n^2 - 1} = \frac{1}{2} \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) = \frac{1}{2} \lim_{N \rightarrow \infty} \sum_{n=2}^N \left(\frac{1}{n-1} - \frac{1}{n+1} \right)$

$$= \frac{1}{2} \lim_{N \rightarrow \infty} \left(1 + \frac{1}{2} - \frac{1}{N+1} \right) = \frac{1}{2} \left(1 + \frac{1}{2} \right) = \frac{3}{4}. \text{ Hence}$$

$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$ is convergent and by comparison test $\sum_{n=2}^{\infty} \frac{n^2}{n^4 - 1}$ is convergent.