
KOÇ UNIVERSITY

MATH 106

SECOND MIDTERM

MAY 9, 2011

Duration of Exam: 100 minutes

INSTRUCTIONS:

- No calculators may be used on the test.
- No books, no notes, no questions, and talking allowed.
- You must always explain your answers and show your work to receive full credit.
- Print (use CAPITAL LETTERS) and sign your name. GOOD LUCK!

SURNAME, Name: _____

Student ID no: _____

Signature: _____

(Check One):
_____ (Kazım Büyükboduk - TTh 11:00-12:15) : _____
_____ (Kazım Büyükboduk - TTh 17:00-18:15) : _____
_____ (Ayşe Soysal - TF 9:30-10:45) : _____
_____ (Barış Coşkunüzer - TTh 9:30-10:45) : _____

PROBLEM	1	2	3	4	5	6	TOTAL
POINTS	20	15	25	15	15	20	110
SCORE							

Problem 1

Let $f(x) = xe^{-x}$, defined for all $x \in \mathbb{R}$.

(a) (3 points)

Find the interval(s) on which $f(x)$ is increasing and the interval(s) on which $f(x)$ is decreasing.

$$f'(x) = e^{-x} - x \cdot e^{-x} = e^{-x}(1-x) \begin{cases} \geq 0 & \text{if } x \leq 1 \\ < 0 & \text{if } x > 1 \end{cases}$$

$\Rightarrow f(x)$ is increasing on $(-\infty, 1]$, decreasing on $(1, \infty)$.

[Arithmetic error in differentiation -1]

(b) (4 points)

Find the critical points of $f(x)$. Classify each of these critical points as a local minimum, local maximum or neither.

Critical points of $f(x)$ are those x for which $f'(x) = 0$ or f is not diff'ble at x . Since f is diff'ble for all $x \in \mathbb{R}$, only critical points are solutions to

$$f'(x) = e^{-x}(1-x) = 0, \text{ namely } x=1.$$

$x=1$ is a local max. because

(i) f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$

or because

(ii) $f''(1) \stackrel{\ominus}{=} -e^{-1} < 0$, so by the second derivative test 1 is a local maximum.

$$f''(x) = -e^{-x}(1-x) - e^{-x}$$

(c) (3 points)

Find the interval(s) on which $f(x)$ is concave up and the intervals on which $f(x)$ is concave down.

$$f''(x) = (e^{-x} - x \cdot e^{-x})' = -e^{-x} - e^{-x} + x \cdot e^{-x} \\ = e^{-x}(-2 + x)$$

$\Rightarrow f$ is concave up on $(2, \infty)$ and
concave down on $(-\infty, 2)$.

[Arithmetic error when differentiating: -1]

(d) (3 points)

Find the inflection points of $f(x)$.

The only inflection point of $f(x)$ is 2, as that is the only point where f changes from being concave up (or down) to concave down (or up).

(e) (3 points)

Find the horizontal asymptotes of $f(x)$.

~~These are the~~

$$y = \lim_{x \rightarrow \infty} x \cdot e^{-x} \stackrel{\text{L'Hospital}}{=} \lim_{x \rightarrow \infty} \frac{1}{e^x} = 0, \text{ and}$$

so the line $y=0$

$$\lim_{x \rightarrow -\infty} \frac{x}{e^x} \stackrel{\text{L'Hospital}}{=} \lim_{x \rightarrow -\infty} x = -\infty, \text{ so no horizontal asymptote as } x \rightarrow -\infty.$$

(f) (4 points)

Sketch the graph of $f(x)$ based on your work in parts (a)-(e).



Max at 1 : +1

Inflection at 2 : +1

$y=0$ as the Horizontal asymptote as $x \rightarrow \infty$: +1

$$\lim_{x \rightarrow -\infty} x \cdot e^{-x} = -\infty \quad : \quad +1$$

Problem 2 (15 points)

Find the point on the curve $y = \sqrt{x}$ closest to the point $(2, 0)$.

+7 Let $d(x) = \sqrt{(\sqrt{x})^2 + (x-2)^2}$, the distance of the point (x, \sqrt{x}) to the point $(2, 0)$.

+3 Then $d(x) = \sqrt{x^2 - 3x + 4}$, and in order to find the closest point to $(2, 0)$ on the curve $y = \sqrt{x}$, we need to find the absolute min. value for the function $d(x)$.

+3
$$d'(x) = \frac{2x-3}{\sqrt{x^2-3x+4}} = 0 \Rightarrow x = 3/2 \text{ is the only critical value.}$$

Furthermore, as $d'(x) > 0$ for $x > 3/2$
 $d'(x) < 0$ for $x < 3/2$,

+2 it follows that d is ~~decreasing~~ ^{decreasing} on ~~(0, 3/2)~~ ~~(0, 3/2)~~,
and ~~decreasing~~ ^{increasing} on $(3/2, \infty)$. This shows that $3/2$
is the absolute min for $d(x)$, so the closest ~~point~~ _{point} to $(2, 0)$ is $(\frac{3}{2}, \sqrt{\frac{3}{2}})$.

(Just ~~checking~~ ^{checking} that $x = 3/2$ is a local min is not enough for credit from the last part)

Problem 3

(a) (6 points)

Evaluate $\int_2^5 x\sqrt{x-1} dx$.

$$\int_2^5 x\sqrt{x-1} dx \stackrel{\ominus}{=} \int_1^4 (u+1)\sqrt{u} du = \int_1^4 (u^{3/2} + u^{1/2}) du =$$

$x-1=u$
 $dx=du$
 $x=u+1$

+4.5

$$= u^{5/2} \cdot \frac{2}{5} + u^{3/2} \cdot \frac{2}{3} \Big|_1^4 = \frac{64}{5} + \frac{16}{3} - \left(\frac{2}{5} + \frac{2}{3} \right)$$

$$= \frac{62}{5} + \frac{14}{3}$$

(b) (6 points)

Evaluate $\int \frac{\ln x}{x^2} dx$.

$$\int \frac{\ln x}{x^2} dx \stackrel{\ominus}{=} \int y \cdot e^{-y} dy \stackrel{\ominus}{=} -e^{-y} \cdot y - \int (-e^{-y}) dy =$$

$\ln x = y$
 $\frac{1}{x} dx = dy$
 $x = e^y$

+2

$u=y \Rightarrow du=dy$
 $e^{-y} dy = dv \Rightarrow v = -e^{-y}$

+3

$$= -e^{-y} \cdot y + \int e^{-y} dy = -e^{-y} y - e^{-y} \quad (+1)$$

$$= -e^{-y} (y+1) \stackrel{\ominus}{=} -\frac{1}{x} \cdot (\ln x + 1)$$

y = ln x

(c) (6 points)

Calculate $\int_{-1}^0 \frac{5x-8}{x^2-3x+2} dx$.

$$\frac{5x-8}{x^2-3x+2} = \frac{5x-8}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}, \text{ where } A, B$$

satisfy $Ax - A + Bx - 2B = 5x - 8 \Leftrightarrow \begin{cases} A+B=5 \\ A+2B=8 \end{cases} \Rightarrow \begin{cases} B=3 \\ A=2 \end{cases}$

$$\Rightarrow \frac{5x-8}{x^2-3x+2} = \frac{2}{x-2} + \frac{3}{x-1} \quad (+4)$$

$$\Rightarrow \int_{-1}^0 \frac{5x-8}{x^2-3x+2} dx = \int_{-1}^0 \frac{2}{x-2} dx + \int_{-1}^0 \frac{3}{x-1} dx = 2 \cdot \ln|x-2| \Big|_{-1}^0 + 3 \ln|x-1| \Big|_{-1}^0$$
$$= 2 \cdot \ln \frac{2}{3} + 3 \cdot \ln 2$$
$$= 5 \ln 2 - \ln 3$$

(d) (7 points)

Evaluate $\frac{d}{dx} \left(\int_{x^2}^{\cos x} e^{\sin x} dx \right)$.

$$\frac{d}{dx} \left(\int_{x^2}^{\cos x} e^{\sin x} dx \right) = \frac{d}{dx} \left(\int_0^{\cos x} e^{\sin x} dx - \int_0^{x^2} e^{\sin x} dx \right) =$$

$$= \frac{d}{dx} \left(F(\cos x) - F(x^2) \right) \quad \text{where } F(u) = \int_0^u e^{\sin x} dx.$$

$$\Leftrightarrow -\sin x \cdot F'(\cos x) - 2x \cdot F'(x^2)$$

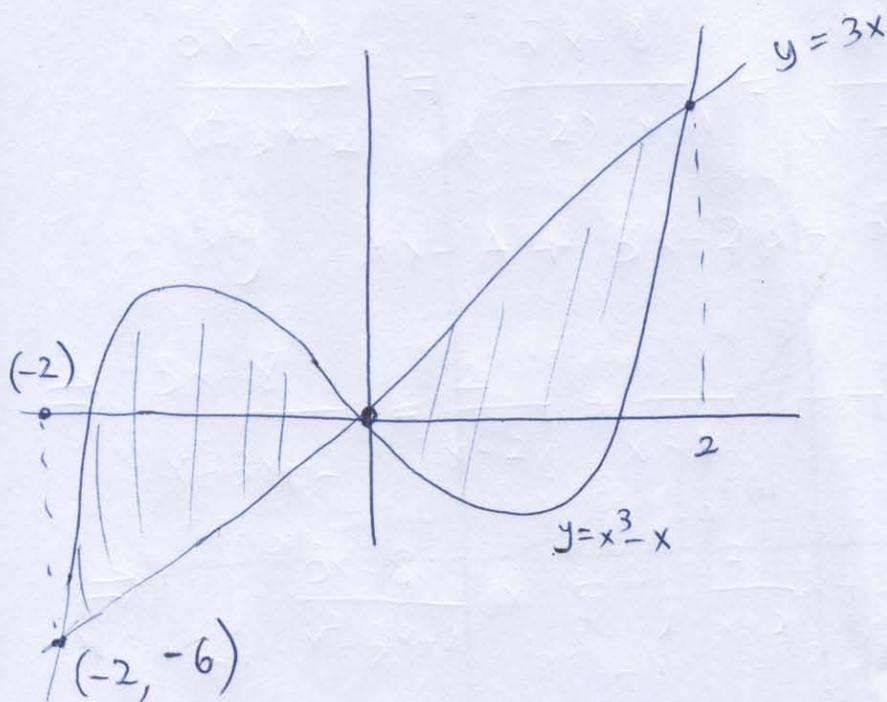
Chain rule

$$\Leftrightarrow -\sin x \cdot e^{\sin(\cos x)} - 2x \cdot e^{\sin(x^2)}$$

FToC: $F'(u) = e^{\sin x}$

Problem 4 (15 points)

Find the area between the curves $y = x^3 - x$ and $y = 3x$.



$$x^3 - x = 3x$$

$$\Leftrightarrow x(x^2 - 4) = 0$$

$$\Leftrightarrow x = 0, 2 \text{ or } -2.$$

missing intersection
point: (-10)

$$\text{Area} = \int_{-2}^0 (x^3 - 4x) dx + \int_0^2 (4x - x^3) dx$$

$$= 2 \cdot \int_0^2 (4x - x^3) dx = 2 \cdot \left(2x^2 - \frac{x^4}{4} \Big|_0^2 \right)$$

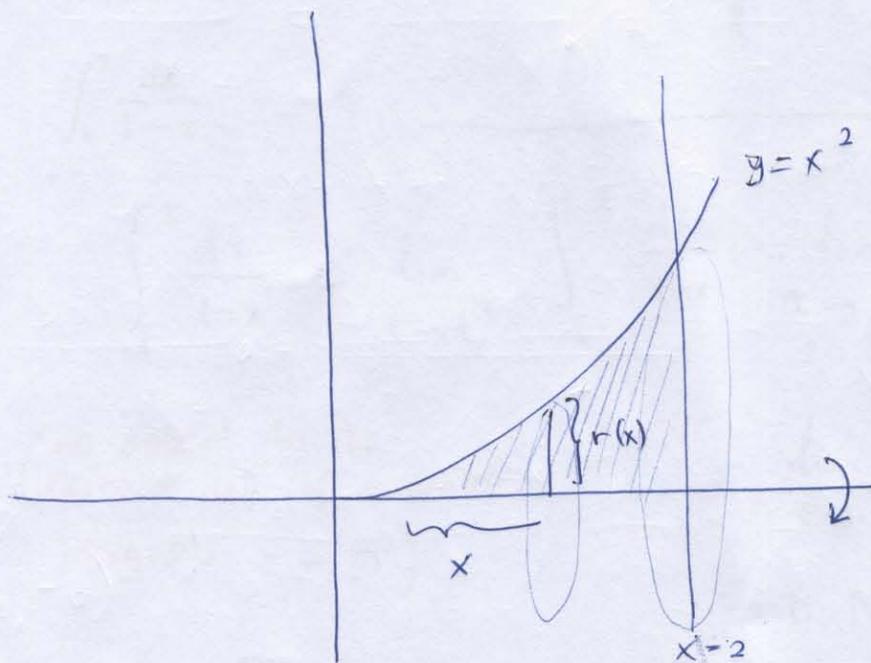
$$= 2 \cdot \frac{16}{3} = \frac{32}{3}$$

[With wrong intersection points, correct integrand]

(+3)

Problem 5 (15 points)

Find the volume of the solid obtained when the region between the graphs of $y = x^2$, $x = 0$, $x = 2$ and $y = 0$ is rotated about the x -axis.



$$\text{Volume} = \int_0^2 \pi r(x)^2 dx \quad (\equiv) \quad \int_0^2 \pi x^4 dx$$

$$r(x) = y = x^2$$

$$= \pi \cdot \frac{x^5}{5} \Big|_0^2 = \frac{32\pi}{5}$$

Problem 6

Determine whether each of the two improper integrals below are convergent or divergent, fully justifying your answer.

(a) (10 points)

$$\int_1^2 \frac{dx}{1-x}$$

Alternatively: $\int_1^2 \frac{dx}{1-x} \in \int_0^1 -\frac{1}{u} du = \text{divergent}$
 by the p-test
 $-1+x=u$
 $dx=du$

$$\int_1^2 \frac{dx}{1-x} = \lim_{t \rightarrow 1^+} \int_t^2 \frac{dx}{1-x} = \lim_{t \rightarrow 1^+} \left[-\ln|1-x| \right]_t^2$$

(no appeal to the correct def. of improper integrals: -5)

$$= \lim_{t \rightarrow 1^+} -\ln|t-1|$$

$$= \text{D.N.E!}$$

So this improper integral is divergent.

(b) (10 points)

$$\int_0^1 \frac{\sin x}{\sqrt[3]{x}} dx$$

for $x \in (0,1)$, $0 \leq \frac{\sin x}{x^{1/3}} \leq \frac{1}{x^{1/3}}$ and

$$\int_0^1 \frac{1}{x^{1/3}} dx = \lim_{t \rightarrow 0^+} \int_t^1 x^{-1/3} dx = \lim_{t \rightarrow 0^+} \left[x^{2/3} \cdot \frac{3}{2} \right]_t^1$$

$$= \frac{3}{2}, \text{ hence } \int_0^1 \frac{1}{x^{1/3}} dx \text{ is convergent.}$$

As $0 \leq \frac{\sin x}{x^{1/3}} \leq \frac{1}{x^{1/3}}$, $\int_0^1 \frac{\sin x}{x^{1/3}} dx$ also converges by the Comparison test.